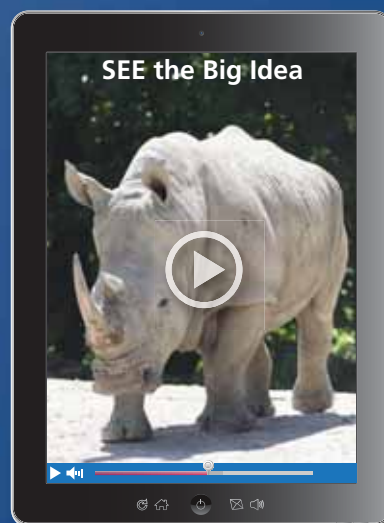


5 Rational Exponents and Radical Functions

- 5.1 n th Roots and Rational Exponents
- 5.2 Properties of Rational Exponents and Radicals
- 5.3 Graphing Radical Functions
- 5.4 Solving Radical Equations and Inequalities
- 5.5 Performing Function Operations
- 5.6 Inverse of a Function



Hull Speed (p. 282)



White Rhino (p. 272)



Concert (p. 268)



Constellations (p. 250)



Mars Rover (p. 254)

Maintaining Mathematical Proficiency

Properties of Integer Exponents

Example 1 Simplify the expression $\frac{x^5 \cdot x^2}{x^3}$.

$$\begin{aligned}\frac{x^5 \cdot x^2}{x^3} &= \frac{x^{5+2}}{x^3} \\ &= \frac{x^7}{x^3} \\ &= x^{7-3} \\ &= x^4\end{aligned}$$

Product of Powers Property

Add exponents.

Quotient of Powers Property

Subtract exponents.

Example 2 Simplify the expression $\left(\frac{2s^3}{t}\right)^2$.

$$\begin{aligned}\left(\frac{2s^3}{t}\right)^2 &= \frac{(2s^3)^2}{t^2} \\ &= \frac{2^2 \cdot (s^3)^2}{t^2} \\ &= \frac{4s^6}{t^2}\end{aligned}$$

Power of a Quotient Property

Power of a Product Property

Power of a Power Property

Simplify the expression.

1. $y^6 \cdot y$

2. $\frac{n^4}{n^3}$

3. $\frac{x^5}{x^6 \cdot x^2}$

4. $\frac{x^6}{x^5} \cdot 3x^2$

5. $\left(\frac{4w^3}{2z^2}\right)^3$

6. $\left(\frac{m^7 \cdot m}{z^2 \cdot m^3}\right)^2$

Rewriting Literal Equations

Example 3 Solve the literal equation $-5y - 2x = 10$ for y .

$$-5y - 2x = 10$$

Write the equation.

$$-5y - 2x + 2x = 10 + 2x$$

Add $2x$ to each side.

$$-5y = 10 + 2x$$

Simplify.

$$\frac{-5y}{-5} = \frac{10 + 2x}{-5}$$

Divide each side by -5 .

$$y = -2 - \frac{2}{5}x$$

Simplify.

Solve the literal equation for y .

7. $4x + y = 2$

8. $x - \frac{1}{3}y = -1$

9. $2y - 9 = 13x$

10. $2xy + 6y = 10$

11. $8x - 4xy = 3$

12. $6x + 7xy = 15$

13. **ABSTRACT REASONING** Is the order in which you apply properties of exponents important? Explain your reasoning.

Mathematical Practices

Mathematically proficient students express numerical answers precisely.

Using Technology to Evaluate Roots

Core Concept

Evaluating Roots with a Calculator

	Example
Square root:	$\sqrt{64} = 8$
Cube root:	$\sqrt[3]{64} = 4$
Fourth root:	$\sqrt[4]{256} = 4$
Fifth root:	$\sqrt[5]{32} = 2$

square root	$\sqrt{(64)}$	8
cube root	${}^3\sqrt{(64)}$	4
fourth root	$4 \times \sqrt{(256)}$	4
fifth root	$5 \times \sqrt{(32)}$	2

EXAMPLE 1 Approximating Roots

Evaluate each root using a calculator. Round your answer to two decimal places.

- a. $\sqrt{50}$ b. $\sqrt[3]{50}$ c. $\sqrt[4]{50}$ d. $\sqrt[5]{50}$

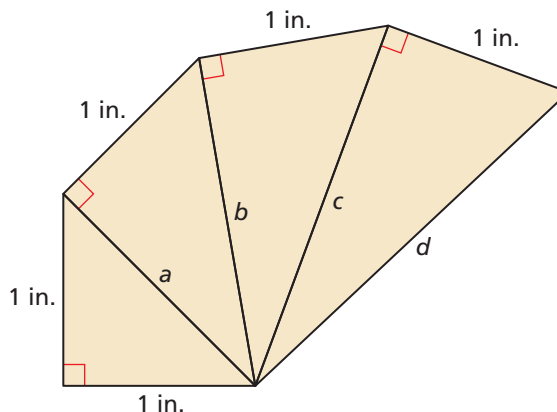
SOLUTION

- a. $\sqrt{50} \approx 7.07$ Round down.
 b. $\sqrt[3]{50} \approx 3.68$ Round down.
 c. $\sqrt[4]{50} \approx 2.66$ Round up.
 d. $\sqrt[5]{50} \approx 2.19$ Round up.

$\sqrt{(50)}$	7.071067812
${}^3\sqrt{(50)}$	3.684031499
$4 \times \sqrt{(50)}$	2.659147948
$5 \times \sqrt{(50)}$	2.186724148

Monitoring Progress

- Use the Pythagorean Theorem to find the exact lengths of a , b , c , and d in the figure.
- Use a calculator to approximate each length to the nearest tenth of an inch.
- Use a ruler to check the reasonableness of your answers.



5.1 n th Roots and Rational Exponents

Essential Question How can you use a rational exponent to represent a power involving a radical?

Previously, you learned that the n th root of a can be represented as

$$\sqrt[n]{a} = a^{1/n} \quad \text{Definition of rational exponent}$$

for any real number a and integer n greater than 1.

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to understand and use stated definitions and previously established results.

EXPLORATION 1 Exploring the Definition of a Rational Exponent

Work with a partner. Use a calculator to show that each statement is true.

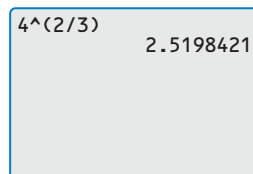
- a. $\sqrt{9} = 9^{1/2}$ b. $\sqrt{2} = 2^{1/2}$ c. $\sqrt[3]{8} = 8^{1/3}$
 d. $\sqrt[3]{3} = 3^{1/3}$ e. $\sqrt[4]{16} = 16^{1/4}$ f. $\sqrt[4]{12} = 12^{1/4}$

EXPLORATION 2 Writing Expressions in Rational Exponent Form

Work with a partner. Use the definition of a rational exponent and the properties of exponents to write each expression as a base with a single rational exponent. Then use a calculator to evaluate each expression. Round your answer to two decimal places.

Sample

$$\begin{aligned} (\sqrt[3]{4})^2 &= (4^{1/3})^2 \\ &= 4^{2/3} \\ &\approx 2.52 \end{aligned}$$



- a. $(\sqrt{5})^3$ b. $(\sqrt[4]{4})^2$ c. $(\sqrt[3]{9})^2$
 d. $(\sqrt[5]{10})^4$ e. $(\sqrt{15})^3$ f. $(\sqrt[3]{27})^4$

EXPLORATION 3 Writing Expressions in Radical Form

Work with a partner. Use the properties of exponents and the definition of a rational exponent to write each expression as a radical raised to an exponent. Then use a calculator to evaluate each expression. Round your answer to two decimal places.

Sample $5^{2/3} = (5^{1/3})^2 = (\sqrt[3]{5})^2 \approx 2.92$

- a. $8^{2/3}$ b. $6^{5/2}$ c. $12^{3/4}$
 d. $10^{3/2}$ e. $16^{3/2}$ f. $20^{6/5}$

Communicate Your Answer

4. How can you use a rational exponent to represent a power involving a radical?
 5. Evaluate each expression *without* using a calculator. Explain your reasoning.
- a. $4^{3/2}$ b. $32^{4/5}$ c. $625^{3/4}$
 d. $49^{3/2}$ e. $125^{4/3}$ f. $100^{6/3}$

5.1 Lesson

Core Vocabulary

n th root of a , p. 238
index of a radical, p. 238

Previous
square root
cube root
exponent

What You Will Learn

- ▶ Find n th roots of numbers.
- ▶ Evaluate expressions with rational exponents.
- ▶ Solve equations using n th roots.

n th Roots

You can extend the concept of a square root to other types of roots. For example, 2 is a cube root of 8 because $2^3 = 8$. In general, for an integer n greater than 1, if $b^n = a$, then b is an **n th root of a** . An n th root of a is written as $\sqrt[n]{a}$, where n is the **index** of the radical.

You can also write an n th root of a as a power of a . If you assume the Power of a Power Property applies to rational exponents, then the following is true.

$$(a^{1/2})^2 = a^{(1/2) \cdot 2} = a^1 = a$$

$$(a^{1/3})^3 = a^{(1/3) \cdot 3} = a^1 = a$$

$$(a^{1/4})^4 = a^{(1/4) \cdot 4} = a^1 = a$$

Because $a^{1/2}$ is a number whose square is a , you can write $\sqrt{a} = a^{1/2}$. Similarly, $\sqrt[3]{a} = a^{1/3}$ and $\sqrt[4]{a} = a^{1/4}$. In general, $\sqrt[n]{a} = a^{1/n}$ for any integer n greater than 1.

Core Concept

Real n th Roots of a

Let n be an integer ($n > 1$) and let a be a real number.

n is an even integer.

$a < 0$ No real n th roots

$a = 0$ One real n th root: $\sqrt[n]{0} = 0$

$a > 0$ Two real n th roots: $\pm\sqrt[n]{a} = \pm a^{1/n}$

n is an odd integer.

$a < 0$ One real n th root: $\sqrt[n]{a} = a^{1/n}$

$a = 0$ One real n th root: $\sqrt[n]{0} = 0$

$a > 0$ One real n th root: $\sqrt[n]{a} = a^{1/n}$

UNDERSTANDING MATHEMATICAL TERMS

When n is even and $a > 0$, there are two real roots. The positive root is called the *principal root*.

EXAMPLE 1 Finding n th Roots

Find the indicated real n th root(s) of a .

a. $n = 3, a = -216$

b. $n = 4, a = 81$

SOLUTION

a. Because $n = 3$ is odd and $a = -216 < 0$, -216 has one real cube root.

Because $(-6)^3 = -216$, you can write $\sqrt[3]{-216} = -6$ or $(-216)^{1/3} = -6$.

b. Because $n = 4$ is even and $a = 81 > 0$, 81 has two real fourth roots.

Because $3^4 = 81$ and $(-3)^4 = 81$, you can write $\pm\sqrt[4]{81} = \pm 3$ or $\pm 81^{1/4} = \pm 3$.

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Find the indicated real n th root(s) of a .

1. $n = 4, a = 16$

2. $n = 2, a = -49$

3. $n = 3, a = -125$

4. $n = 5, a = 243$

Rational Exponents

A rational exponent does not have to be of the form $\frac{1}{n}$. Other rational numbers, such as $\frac{3}{2}$ and $-\frac{1}{2}$, can also be used as exponents. Two properties of rational exponents are shown below.

Core Concept

Rational Exponents

Let $a^{1/n}$ be an n th root of a , and let m be a positive integer.

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$$

$$a^{-m/n} = \frac{1}{a^{m/n}} = \frac{1}{(a^{1/n})^m} = \frac{1}{(\sqrt[n]{a})^m}, a \neq 0$$

EXAMPLE 2 Evaluating Expressions with Rational Exponents

Evaluate each expression.

a. $16^{3/2}$

b. $32^{-3/5}$

SOLUTION

Rational Exponent Form

a. $16^{3/2} = (16^{1/2})^3 = 4^3 = 64$

b. $32^{-3/5} = \frac{1}{32^{3/5}} = \frac{1}{(32^{1/5})^3} = \frac{1}{2^3} = \frac{1}{8}$

Radical Form

$$16^{3/2} = (\sqrt{16})^3 = 4^3 = 64$$

$$32^{-3/5} = \frac{1}{32^{3/5}} = \frac{1}{(\sqrt[5]{32})^3} = \frac{1}{2^3} = \frac{1}{8}$$

When using a calculator to approximate an n th root, you may want to rewrite the n th root in rational exponent form.

EXAMPLE 3 Approximating Expressions with Rational Exponents

Evaluate each expression using a calculator. Round your answer to two decimal places.

a. $9^{1/5}$

b. $12^{3/8}$

c. $(\sqrt[4]{7})^3$

SOLUTION

a. $9^{1/5} \approx 1.55$

b. $12^{3/8} \approx 2.54$

c. Before evaluating $(\sqrt[4]{7})^3$, rewrite the expression in rational exponent form.

$$(\sqrt[4]{7})^3 = 7^{3/4} \approx 4.30$$

$9^{1/5}$	1.551845574
$12^{3/8}$	2.539176951
$7^{3/4}$	4.303517071

COMMON ERROR

Be sure to use parentheses to enclose a rational exponent: $9^{1/5} \approx 1.55$. Without them, the calculator evaluates a power and then divides: $9^{1/5} = 1.8$.

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Evaluate the expression without using a calculator.

5. $4^{5/2}$

6. $9^{-1/2}$

7. $81^{3/4}$

8. $1^{7/8}$

Evaluate the expression using a calculator. Round your answer to two decimal places when appropriate.

9. $6^{2/5}$

10. $64^{-2/3}$

11. $(\sqrt[4]{16})^5$

12. $(\sqrt[3]{-30})^2$

Solving Equations Using n th Roots

To solve an equation of the form $u^n = d$, where u is an algebraic expression, take the n th root of each side.

EXAMPLE 4 Solving Equations Using n th Roots

Find the real solution(s) of (a) $4x^5 = 128$ and (b) $(x - 3)^4 = 21$.

SOLUTION

a. $4x^5 = 128$

$$x^5 = 32$$

$$x = \sqrt[5]{32}$$

$$x = 2$$

Write original equation.

Divide each side by 4.

Take fifth root of each side.

Simplify.

▶ The solution is $x = 2$.

b. $(x - 3)^4 = 21$

$$x - 3 = \pm \sqrt[4]{21}$$

$$x = 3 \pm \sqrt[4]{21}$$

$$x = 3 + \sqrt[4]{21} \text{ or } x = 3 - \sqrt[4]{21}$$

$$x \approx 5.14 \quad \text{or} \quad x \approx 0.86$$

Write original equation.

Take fourth root of each side.

Add 3 to each side.

Write solutions separately.

Use a calculator.

▶ The solutions are $x \approx 5.14$ and $x \approx 0.86$.

COMMON ERROR

When n is even and $a > 0$, be sure to consider both the positive and negative n th roots of a .

EXAMPLE 5 Real-Life Application

A hospital purchases an ultrasound machine for \$50,000. The hospital expects the useful life of the machine to be 10 years, at which time its value will have depreciated to \$8000. The hospital uses the declining balances method for depreciation, so the annual depreciation rate r (in decimal form) is given by the formula

$$r = 1 - \left(\frac{S}{C}\right)^{1/n}$$

In the formula, n is the useful life of the item (in years), S is the salvage value (in dollars), and C is the original cost (in dollars). What annual depreciation rate did the hospital use?

SOLUTION

The useful life is 10 years, so $n = 10$. The machine depreciates to \$8000, so $S = 8000$. The original cost is \$50,000, so $C = 50,000$. So, the annual depreciation rate is

$$r = 1 - \left(\frac{S}{C}\right)^{1/n} = 1 - \left(\frac{8000}{50,000}\right)^{1/10} = 1 - \left(\frac{4}{25}\right)^{1/10} \approx 0.167.$$

▶ The annual depreciation rate is about 0.167, or 16.7%.

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Find the real solution(s) of the equation. Round your answer to two decimal places when appropriate.

13. $8x^3 = 64$ 14. $\frac{1}{2}x^5 = 512$ 15. $(x + 5)^4 = 16$ 16. $(x - 2)^3 = -14$

17. **WHAT IF?** In Example 5, what is the annual depreciation rate when the salvage value is \$6000?



5.1 Exercises

Dynamic Solutions available at BigIdeasMath.com

Vocabulary and Core Concept Check

- VOCABULARY** Rewrite the expression $a^{-s/t}$ in radical form. Then state the index of the radical.
- COMPLETE THE SENTENCE** For an integer n greater than 1, if $b^n = a$, then b is a(n) _____ of a .
- WRITING** Explain how to use the sign of a to determine the number of real fourth roots of a and the number of real fifth roots of a .
- WHICH ONE DOESN'T BELONG?** Which expression does *not* belong with the other three? Explain your reasoning.

$$(a^{1/n})^m$$

$$(\sqrt[n]{a})^m$$

$$(\sqrt[m]{a})^{-n}$$

$$a^{m/n}$$

Monitoring Progress and Modeling with Mathematics


In Exercises 5–10, find the indicated real n th root(s) of a . (See Example 1.)


- | | |
|---------------------|-----------------------|
| 5. $n = 3, a = 8$ | 6. $n = 5, a = -1$ |
| 7. $n = 2, a = 0$ | 8. $n = 4, a = 256$ |
| 9. $n = 5, a = -32$ | 10. $n = 6, a = -729$ |

In Exercises 11–18, evaluate the expression without using a calculator. (See Example 2.)

- | | |
|--------------------|-------------------|
| 11. $64^{1/6}$ | 12. $8^{1/3}$ |
| 13. $25^{3/2}$ | 14. $81^{3/4}$ |
| 15. $(-243)^{1/5}$ | 16. $(-64)^{4/3}$ |
| 17. $8^{-2/3}$ | 18. $16^{-7/4}$ |

ERROR ANALYSIS In Exercises 19 and 20, describe and correct the error in evaluating the expression.

19.  $27^{2/3} = (27^{1/3})^2$
 $= 9^2$
 $= 81$

20.  $256^{4/3} = (\sqrt[4]{256})^3$
 $= 4^3$
 $= 64$

USING STRUCTURE In Exercises 21–24, match the equivalent expressions. Explain your reasoning.

- | | |
|-----------------------------|---------------|
| 21. $(\sqrt[3]{5})^4$ | A. $5^{-1/4}$ |
| 22. $(\sqrt[4]{5})^3$ | B. $5^{4/3}$ |
| 23. $\frac{1}{\sqrt[4]{5}}$ | C. $-5^{1/4}$ |
| 24. $-\sqrt[4]{5}$ | D. $5^{3/4}$ |

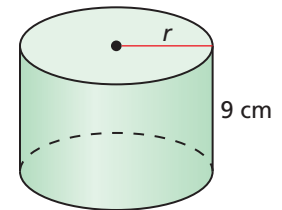
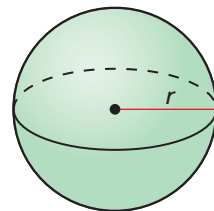
In Exercises 25–32, evaluate the expression using a calculator. Round your answer to two decimal places when appropriate. (See Example 3.)

- | | |
|-------------------------|------------------------|
| 25. $\sqrt[5]{32,768}$ | 26. $\sqrt[7]{1695}$ |
| 27. $25^{-1/3}$ | 28. $85^{1/6}$ |
| 29. $20,736^{4/5}$ | 30. $86^{-5/6}$ |
| 31. $(\sqrt[4]{187})^3$ | 32. $(\sqrt[5]{-8})^8$ |

MATHEMATICAL CONNECTIONS In Exercises 33 and 34, find the radius of the figure with the given volume.

33. $V = 216 \text{ ft}^3$

34. $V = 1332 \text{ cm}^3$



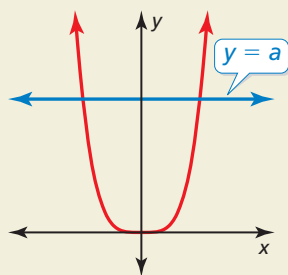
In Exercises 35–44, find the real solution(s) of the equation. Round your answer to two decimal places when appropriate. (See Example 4.)

35. $x^3 = 125$ 36. $5x^3 = 1080$
 37. $(x + 10)^5 = 70$ 38. $(x - 5)^4 = 256$
 39. $x^5 = -48$ 40. $7x^4 = 56$
 41. $x^6 + 36 = 100$ 42. $x^3 + 40 = 25$
 43. $\frac{1}{3}x^4 = 27$ 44. $\frac{1}{6}x^3 = -36$

45. **MODELING WITH MATHEMATICS** When the average price of an item increases from p_1 to p_2 over a period of n years, the annual rate of inflation r (in decimal form) is given by $r = \left(\frac{p_2}{p_1}\right)^{1/n} - 1$. Find the rate of inflation for each item in the table. (See Example 5.)

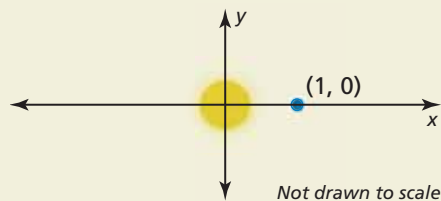
Item	Price in 1913	Price in 2013
Potatoes (lb)	\$0.016	\$0.627
Ham (lb)	\$0.251	\$2.693
Eggs (dozen)	\$0.373	\$1.933

46. **HOW DO YOU SEE IT?** The graph of $y = x^n$ is shown in red. What can you conclude about the value of n ? Determine the number of real n th roots of a . Explain your reasoning.

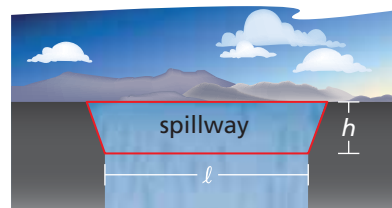


47. **NUMBER SENSE** Between which two consecutive integers does $\sqrt[4]{125}$ lie? Explain your reasoning.

48. **THOUGHT PROVOKING** In 1619, Johannes Kepler published his third law, which can be given by $d^3 = t^2$, where d is the mean distance (in astronomical units) of a planet from the Sun and t is the time (in years) it takes the planet to orbit the Sun. It takes Mars 1.88 years to orbit the Sun. Graph a possible location of Mars. Justify your answer. (The diagram shows the Sun at the origin of the xy -plane and a possible location of Earth.)



49. **PROBLEM SOLVING** A weir is a dam that is built across a river to regulate the flow of water. The flow rate Q (in cubic feet per second) can be calculated using the formula $Q = 3.367\ell h^{3/2}$, where ℓ is the length (in feet) of the bottom of the spillway and h is the depth (in feet) of the water on the spillway. Determine the flow rate of a weir with a spillway that is 20 feet long and has a water depth of 5 feet.



50. **REPEATED REASONING** The mass of the particles that a river can transport is proportional to the sixth power of the speed of the river. A certain river normally flows at a speed of 1 meter per second. What must its speed be in order to transport particles that are twice as massive as usual? 10 times as massive? 100 times as massive?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. Write your answer using only positive exponents. (Skills Review Handbook)

51. $5 \cdot 5^4$ 52. $\frac{4^2}{4^7}$ 53. $(z^2)^{-3}$ 54. $\left(\frac{3x}{2}\right)^4$

Write the number in standard form. (Skills Review Handbook)

55. 5×10^3 56. 4×10^{-2}
 57. 8.2×10^{-1} 58. 6.93×10^6

5.2 Properties of Rational Exponents and Radicals

Essential Question How can you use properties of exponents to simplify products and quotients of radicals?

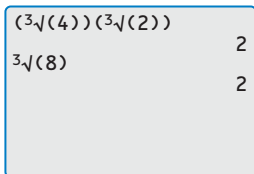
EXPLORATION 1 Reviewing Properties of Exponents

Work with a partner. Let a and b be real numbers. Use the properties of exponents to complete each statement. Then match each completed statement with the property it illustrates.

Statement	Property
a. $a^{-2} = \underline{\hspace{2cm}}, a \neq 0$	A. Product of Powers
b. $(ab)^4 = \underline{\hspace{2cm}}$	B. Power of a Power
c. $(a^3)^4 = \underline{\hspace{2cm}}$	C. Power of a Product
d. $a^3 \cdot a^4 = \underline{\hspace{2cm}}$	D. Negative Exponent
e. $\left(\frac{a}{b}\right)^3 = \underline{\hspace{2cm}}, b \neq 0$	E. Zero Exponent
f. $\frac{a^6}{a^2} = \underline{\hspace{2cm}}, a \neq 0$	F. Quotient of Powers
g. $a^0 = \underline{\hspace{2cm}}, a \neq 0$	G. Power of a Quotient

USING TOOLS STRATEGICALLY

To be proficient in math, you need to consider the tools available to help you check your answers. For instance, the following calculator screen shows that $\sqrt[3]{4} \cdot \sqrt[3]{2}$ and $\sqrt[3]{8}$ are equivalent.



EXPLORATION 2 Simplifying Expressions with Rational Exponents

Work with a partner. Show that you can apply the properties of integer exponents to rational exponents by simplifying each expression. Use a calculator to check your answers.

- | | | |
|----------------------------|------------------------------|------------------------------|
| a. $5^{2/3} \cdot 5^{4/3}$ | b. $3^{1/5} \cdot 3^{4/5}$ | c. $(4^{2/3})^3$ |
| d. $(10^{1/2})^4$ | e. $\frac{8^{5/2}}{8^{1/2}}$ | f. $\frac{7^{2/3}}{7^{5/3}}$ |

EXPLORATION 3 Simplifying Products and Quotients of Radicals

Work with a partner. Use the properties of exponents to write each expression as a single radical. Then evaluate each expression. Use a calculator to check your answers.

- | | | |
|---------------------------------|---|--|
| a. $\sqrt{3} \cdot \sqrt{12}$ | b. $\sqrt[3]{5} \cdot \sqrt[3]{25}$ | c. $\sqrt[4]{27} \cdot \sqrt[4]{3}$ |
| d. $\frac{\sqrt{98}}{\sqrt{2}}$ | e. $\frac{\sqrt[4]{4}}{\sqrt[4]{1024}}$ | f. $\frac{\sqrt[3]{625}}{\sqrt[3]{5}}$ |

Communicate Your Answer

4. How can you use properties of exponents to simplify products and quotients of radicals?

5. Simplify each expression.

- | | | |
|-------------------------------|---|---------------------------------|
| a. $\sqrt{27} \cdot \sqrt{6}$ | b. $\frac{\sqrt[3]{240}}{\sqrt[3]{15}}$ | c. $(5^{1/2} \cdot 16^{1/4})^2$ |
|-------------------------------|---|---------------------------------|

5.2 Lesson

Core Vocabulary

simplest form of a radical,
p. 245

conjugate, p. 246

like radicals, p. 246

Previous

properties of integer
exponents

rationalizing the
denominator

absolute value

COMMON ERROR

When you multiply powers, do *not* multiply the exponents. For example, $3^2 \cdot 3^5 \neq 3^{10}$.

What You Will Learn

- ▶ Use properties of rational exponents to simplify expressions with rational exponents.
- ▶ Use properties of radicals to simplify and write radical expressions in simplest form.

Properties of Rational Exponents

The properties of integer exponents that you have previously learned can also be applied to rational exponents.

Core Concept

Properties of Rational Exponents

Let a and b be real numbers and let m and n be rational numbers, such that the quantities in each property are real numbers.

Property Name	Definition	Example
Product of Powers	$a^m \cdot a^n = a^{m+n}$	$5^{1/2} \cdot 5^{3/2} = 5^{(1/2+3/2)} = 5^2 = 25$
Power of a Power	$(a^m)^n = a^{mn}$	$(3^{5/2})^2 = 3^{(5/2 \cdot 2)} = 3^5 = 243$
Power of a Product	$(ab)^m = a^m b^m$	$(16 \cdot 9)^{1/2} = 16^{1/2} \cdot 9^{1/2} = 4 \cdot 3 = 12$
Negative Exponent	$a^{-m} = \frac{1}{a^m}, a \neq 0$	$36^{-1/2} = \frac{1}{36^{1/2}} = \frac{1}{6}$
Zero Exponent	$a^0 = 1, a \neq 0$	$213^0 = 1$
Quotient of Powers	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$	$\frac{4^{5/2}}{4^{1/2}} = 4^{(5/2-1/2)} = 4^2 = 16$
Power of a Quotient	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$	$\left(\frac{27}{64}\right)^{1/3} = \frac{27^{1/3}}{64^{1/3}} = \frac{3}{4}$

EXAMPLE 1 Using Properties of Exponents

Use the properties of rational exponents to simplify each expression.

- $7^{1/4} \cdot 7^{1/2} = 7^{(1/4+1/2)} = 7^{3/4}$
- $(6^{1/2} \cdot 4^{1/3})^2 = (6^{1/2})^2 \cdot (4^{1/3})^2 = 6^{(1/2 \cdot 2)} \cdot 4^{(1/3 \cdot 2)} = 6^1 \cdot 4^{2/3} = 6 \cdot 4^{2/3}$
- $(4^5 \cdot 3^5)^{-1/5} = [(4 \cdot 3)^5]^{-1/5} = (12^5)^{-1/5} = 12^{[5 \cdot (-1/5)]} = 12^{-1} = \frac{1}{12}$
- $\frac{5}{5^{1/3}} = \frac{5^1}{5^{1/3}} = 5^{(1-1/3)} = 5^{2/3}$
- $\left(\frac{42^{1/3}}{6^{1/3}}\right)^2 = \left[\left(\frac{42}{6}\right)^{1/3}\right]^2 = (7^{1/3})^2 = 7^{(1/3 \cdot 2)} = 7^{2/3}$

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Simplify the expression.

- $2^{3/4} \cdot 2^{1/2}$
- $\frac{3}{3^{1/4}}$
- $\left(\frac{20^{1/2}}{5^{1/2}}\right)^3$
- $(5^{1/3} \cdot 7^{1/4})^3$

Simplifying Radical Expressions

The Power of a Product and Power of a Quotient properties can be expressed using radical notation when $m = \frac{1}{n}$ for some integer n greater than 1.

Core Concept

Properties of Radicals

Let a and b be real numbers and let n be an integer greater than 1.

Property Name	Definition	Example
Product Property	$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$	$\sqrt[3]{4} \cdot \sqrt[3]{2} = \sqrt[3]{8} = 2$
Quotient Property	$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, b \neq 0$	$\frac{\sqrt[4]{162}}{\sqrt[4]{2}} = \sqrt[4]{\frac{162}{2}} = \sqrt[4]{81} = 3$

EXAMPLE 2 Using Properties of Radicals

Use the properties of radicals to simplify each expression.

a. $\sqrt[3]{12} \cdot \sqrt[3]{18} = \sqrt[3]{12 \cdot 18} = \sqrt[3]{216} = 6$ Product Property of Radicals

b. $\frac{\sqrt[4]{80}}{\sqrt[4]{5}} = \sqrt[4]{\frac{80}{5}} = \sqrt[4]{16} = 2$ Quotient Property of Radicals

An expression involving a radical with index n is in **simplest form** when these three conditions are met.

- No radicands have perfect n th powers as factors other than 1.
- No radicands contain fractions.
- No radicals appear in the denominator of a fraction.

To meet the last two conditions, rationalize the denominator by multiplying the expression by an appropriate form of 1 that eliminates the radical from the denominator.

EXAMPLE 3 Writing Radicals in Simplest Form

Write each expression in simplest form.

a. $\sqrt[3]{135}$ b. $\frac{\sqrt[5]{7}}{\sqrt[5]{8}}$

SOLUTION

a. $\sqrt[3]{135} = \sqrt[3]{27 \cdot 5}$ Factor out perfect cube.
 $= \sqrt[3]{27} \cdot \sqrt[3]{5}$ Product Property of Radicals
 $= 3\sqrt[3]{5}$ Simplify.

b. $\frac{\sqrt[5]{7}}{\sqrt[5]{8}} = \frac{\sqrt[5]{7}}{\sqrt[5]{8}} \cdot \frac{\sqrt[5]{4}}{\sqrt[5]{4}}$ Make the radicand in the denominator a perfect fifth power.
 $= \frac{\sqrt[5]{28}}{\sqrt[5]{32}}$ Product Property of Radicals
 $= \frac{\sqrt[5]{28}}{2}$ Simplify.

For a denominator that is a sum or difference involving square roots, multiply both the numerator and denominator by the **conjugate** of the denominator. The expressions

$$a\sqrt{b} + c\sqrt{d} \quad \text{and} \quad a\sqrt{b} - c\sqrt{d}$$

are conjugates of each other, where a , b , c , and d are rational numbers.

EXAMPLE 4 Writing a Radical Expression in Simplest Form

Write $\frac{1}{5 + \sqrt{3}}$ in simplest form.

SOLUTION

$$\begin{aligned} \frac{1}{5 + \sqrt{3}} &= \frac{1}{5 + \sqrt{3}} \cdot \frac{5 - \sqrt{3}}{5 - \sqrt{3}} && \text{The conjugate of } 5 + \sqrt{3} \text{ is } 5 - \sqrt{3}. \\ &= \frac{1(5 - \sqrt{3})}{5^2 - (\sqrt{3})^2} && \text{Sum and Difference Pattern} \\ &= \frac{5 - \sqrt{3}}{22} && \text{Simplify.} \end{aligned}$$

Radical expressions with the same index and radicand are **like radicals**. To add or subtract like radicals, use the Distributive Property.

EXAMPLE 5 Adding and Subtracting Like Radicals and Roots

Simplify each expression.

a. $\sqrt[4]{10} + 7\sqrt[4]{10}$ b. $2(8^{1/5}) + 10(8^{1/5})$ c. $\sqrt[3]{54} - \sqrt[3]{2}$

SOLUTION

a. $\sqrt[4]{10} + 7\sqrt[4]{10} = (1 + 7)\sqrt[4]{10} = 8\sqrt[4]{10}$
 b. $2(8^{1/5}) + 10(8^{1/5}) = (2 + 10)(8^{1/5}) = 12(8^{1/5})$
 c. $\sqrt[3]{54} - \sqrt[3]{2} = \sqrt[3]{27} \cdot \sqrt[3]{2} - \sqrt[3]{2} = 3\sqrt[3]{2} - \sqrt[3]{2} = (3 - 1)\sqrt[3]{2} = 2\sqrt[3]{2}$

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Simplify the expression.

5. $\sqrt[4]{27} \cdot \sqrt[4]{3}$ 6. $\frac{\sqrt[3]{250}}{\sqrt[3]{2}}$ 7. $\sqrt[3]{104}$ 8. $\sqrt[5]{\frac{3}{4}}$
 9. $\frac{3}{6 - \sqrt{2}}$ 10. $7\sqrt[5]{12} - \sqrt[5]{12}$ 11. $4(9^{2/3}) + 8(9^{2/3})$ 12. $\sqrt[3]{5} + \sqrt[3]{40}$

The properties of rational exponents and radicals can also be applied to expressions involving variables. Because a variable can be positive, negative, or zero, sometimes absolute value is needed when simplifying a variable expression.

	Rule	Example
When n is odd	$\sqrt[n]{x^n} = x$	$\sqrt[7]{5^7} = 5$ and $\sqrt[7]{(-5)^7} = -5$
When n is even	$\sqrt[n]{x^n} = x $	$\sqrt[4]{3^4} = 3$ and $\sqrt[4]{(-3)^4} = 3$

Absolute value is not needed when all variables are assumed to be positive.

EXAMPLE 6 Simplifying Variable Expressions

Simplify each expression.

a. $\sqrt[3]{64y^6}$

b. $\sqrt[4]{\frac{x^4}{y^8}}$

STUDY TIP

You do not need to take the absolute value of y because y is being squared.

SOLUTION

a. $\sqrt[3]{64y^6} = \sqrt[3]{4^3(y^2)^3} = \sqrt[3]{4^3} \cdot \sqrt[3]{(y^2)^3} = 4y^2$

b. $\sqrt[4]{\frac{x^4}{y^8}} = \frac{\sqrt[4]{x^4}}{\sqrt[4]{y^8}} = \frac{\sqrt[4]{x^4}}{\sqrt[4]{(y^2)^4}} = \frac{|x|}{y^2}$

EXAMPLE 7 Writing Variable Expressions in Simplest Form

Write each expression in simplest form. Assume all variables are positive.

a. $\sqrt[5]{4a^8b^{14}c^5}$

b. $\frac{x}{\sqrt[3]{y^8}}$

c. $\frac{14xy^{1/3}}{2x^{3/4}z^{-6}}$

SOLUTION

a. $\sqrt[5]{4a^8b^{14}c^5} = \sqrt[5]{4a^5a^3b^{10}b^4c^5}$
 $= \sqrt[5]{a^5b^{10}c^5} \cdot \sqrt[5]{4a^3b^4}$
 $= ab^2c\sqrt[5]{4a^3b^4}$

Factor out perfect fifth powers.

Product Property of Radicals

Simplify.

b. $\frac{x}{\sqrt[3]{y^8}} = \frac{x}{\sqrt[3]{y^8}} \cdot \frac{\sqrt[3]{y}}{\sqrt[3]{y}}$
 $= \frac{x\sqrt[3]{y}}{\sqrt[3]{y^9}}$
 $= \frac{x\sqrt[3]{y}}{y^3}$

Make denominator a perfect cube.

Product Property of Radicals

Simplify.

c. $\frac{14xy^{1/3}}{2x^{3/4}z^{-6}} = 7x^{(1-3/4)}y^{1/3}z^{-(-6)} = 7x^{1/4}y^{1/3}z^6$

EXAMPLE 8 Adding and Subtracting Variable Expressions

Perform each indicated operation. Assume all variables are positive.

a. $5\sqrt{y} + 6\sqrt{y}$

b. $12\sqrt[3]{2z^5} - z\sqrt[3]{54z^2}$

SOLUTION

a. $5\sqrt{y} + 6\sqrt{y} = (5 + 6)\sqrt{y} = 11\sqrt{y}$

b. $12\sqrt[3]{2z^5} - z\sqrt[3]{54z^2} = 12z\sqrt[3]{2z^2} - 3z\sqrt[3]{2z^2} = (12z - 3z)\sqrt[3]{2z^2} = 9z\sqrt[3]{2z^2}$

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Simplify the expression. Assume all variables are positive.

13. $\sqrt[3]{27q^9}$

14. $\sqrt{\frac{x^{10}}{y^5}}$

15. $\frac{6xy^{3/4}}{3x^{1/2}y^{1/2}}$

16. $\sqrt{9w^5} - w\sqrt{w^3}$

Vocabulary and Core Concept Check

- WRITING** How do you know when a radical expression is in simplest form?
- WHICH ONE DOESN'T BELONG?** Which radical expression does *not* belong with the other three? Explain your reasoning.

$$\sqrt[3]{\frac{4}{5}}$$

$$2\sqrt{x}$$

$$\sqrt[4]{11}$$

$$3\sqrt[5]{9x}$$

Monitoring Progress and Modeling with Mathematics

In Exercises 3–12, use the properties of rational exponents to simplify the expression. (See Example 1.)

- | | |
|--|---|
| 3. $(9^2)^{1/3}$ | 4. $(12^2)^{1/4}$ |
| 5. $\frac{6}{6^{1/4}}$ | 6. $\frac{7}{7^{1/3}}$ |
| 7. $\left(\frac{8^4}{10^4}\right)^{-1/4}$ | 8. $\left(\frac{9^3}{6^3}\right)^{-1/3}$ |
| 9. $(3^{-2/3} \cdot 3^{1/3})^{-1}$ | 10. $(5^{1/2} \cdot 5^{-3/2})^{-1/4}$ |
| 11. $\frac{2^{2/3} \cdot 16^{2/3}}{4^{2/3}}$ | 12. $\frac{49^{3/8} \cdot 49^{7/8}}{7^{5/4}}$ |

In Exercises 13–20, use the properties of radicals to simplify the expression. (See Example 2.)

- | | |
|--|---|
| 13. $\sqrt{2} \cdot \sqrt{72}$ | 14. $\sqrt[3]{16} \cdot \sqrt[3]{32}$ |
| 15. $\sqrt[4]{6} \cdot \sqrt[4]{8}$ | 16. $\sqrt[4]{8} \cdot \sqrt[4]{8}$ |
| 17. $\frac{\sqrt[5]{486}}{\sqrt{2}}$ | 18. $\frac{\sqrt{2}}{\sqrt{32}}$ |
| 19. $\frac{\sqrt[3]{6} \cdot \sqrt[3]{72}}{\sqrt[3]{2}}$ | 20. $\frac{\sqrt[3]{3} \cdot \sqrt[3]{18}}{\sqrt{2} \cdot \sqrt[6]{2}}$ |

In Exercises 21–28, write the expression in simplest form. (See Example 3.)

- | | |
|---------------------------------------|--|
| 21. $\sqrt[4]{567}$ | 22. $\sqrt[5]{288}$ |
| 23. $\frac{\sqrt[3]{5}}{\sqrt[3]{4}}$ | 24. $\frac{\sqrt[4]{4}}{\sqrt[4]{27}}$ |
| 25. $\sqrt{\frac{3}{8}}$ | 26. $\sqrt[3]{\frac{7}{4}}$ |
| 27. $\sqrt[3]{\frac{64}{49}}$ | 28. $\sqrt[4]{\frac{1296}{25}}$ |

In Exercises 29–36, write the expression in simplest form. (See Example 4.)

- | | |
|--|---|
| 29. $\frac{1}{1 + \sqrt{3}}$ | 30. $\frac{1}{2 + \sqrt{5}}$ |
| 31. $\frac{5}{3 - \sqrt{2}}$ | 32. $\frac{11}{9 - \sqrt{6}}$ |
| 33. $\frac{9}{\sqrt{3} + \sqrt{7}}$ | 34. $\frac{2}{\sqrt{8} + \sqrt{7}}$ |
| 35. $\frac{\sqrt{6}}{\sqrt{3} - \sqrt{5}}$ | 36. $\frac{\sqrt{7}}{\sqrt{10} - \sqrt{2}}$ |

In Exercises 37–46, simplify the expression. (See Example 5.)

- | | |
|-------------------------------------|------------------------------------|
| 37. $9\sqrt[3]{11} + 3\sqrt[3]{11}$ | 38. $8\sqrt[6]{5} - 12\sqrt[6]{5}$ |
| 39. $3(11^{1/4}) + 9(11^{1/4})$ | 40. $13(8^{3/4}) - 4(8^{3/4})$ |
| 41. $5\sqrt{12} - 19\sqrt{3}$ | 42. $27\sqrt{6} + 7\sqrt{150}$ |
| 43. $\sqrt[5]{224} + 3\sqrt[5]{7}$ | 44. $7\sqrt[3]{2} - \sqrt[3]{128}$ |
| 45. $5(24^{1/3}) - 4(3^{1/3})$ | 46. $5^{1/4} + 6(405^{1/4})$ |

47. **ERROR ANALYSIS** Describe and correct the error in simplifying the expression.

$$\begin{aligned}
 \times \quad 3\sqrt[3]{12} + 5\sqrt[3]{12} &= (3 + 5)\sqrt[3]{24} \\
 &= 8\sqrt[3]{24} \\
 &= 8\sqrt[3]{8 \cdot 3} \\
 &= 8 \cdot 2\sqrt[3]{3} \\
 &= 16\sqrt[3]{3}
 \end{aligned}$$

48. **MULTIPLE REPRESENTATIONS** Which radical expressions are like radicals?

(A) $(5^{2/9})^{3/2}$ (B) $\frac{5^3}{(\sqrt[3]{5})^8}$
 (C) $\sqrt[3]{625}$ (D) $\sqrt[3]{5145} - \sqrt[3]{875}$
 (E) $\sqrt[3]{5} + 3\sqrt[3]{5}$ (F) $7\sqrt[4]{80} - 2\sqrt[4]{405}$

In Exercises 49–54, simplify the expression.
 (See Example 6.)

49. $\sqrt[4]{81y^8}$ 50. $\sqrt[3]{64r^3t^6}$
 51. $\sqrt[5]{\frac{m^{10}}{n^5}}$ 52. $\sqrt[4]{\frac{k^{16}}{16z^4}}$
 53. $\sqrt[6]{\frac{g^6h}{h^7}}$ 54. $\sqrt[8]{\frac{n^{18}p^7}{n^2p^{-1}}}$

55. **ERROR ANALYSIS** Describe and correct the error in simplifying the expression.

X

$$\begin{aligned}\sqrt[6]{\frac{64h^{12}}{g^6}} &= \frac{\sqrt[6]{64h^{12}}}{\sqrt[6]{g^6}} \\ &= \frac{\sqrt[6]{2^6 \cdot (h^2)^6}}{\sqrt[6]{g^6}} \\ &= \frac{2h^2}{g}\end{aligned}$$

56. **OPEN-ENDED** Write two variable expressions involving radicals, one that needs absolute value in simplifying and one that does not need absolute value. Justify your answers.

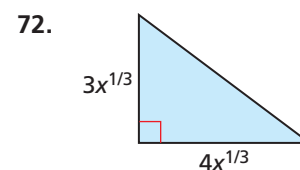
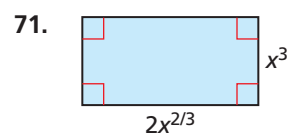
In Exercises 57–64, write the expression in simplest form. Assume all variables are positive. (See Example 7.)

57. $\sqrt{81a^7b^{12}c^9}$ 58. $\sqrt[3]{125r^4s^9t^7}$
 59. $\sqrt[5]{\frac{160m^6}{n^7}}$ 60. $\sqrt[4]{\frac{405x^3y^3}{5x^{-1}y}}$
 61. $\frac{\sqrt[3]{w} \cdot \sqrt{w^5}}{\sqrt{25w^{16}}}$ 62. $\frac{\sqrt[4]{v^6}}{\sqrt[7]{v^5}}$
 63. $\frac{18w^{1/3}v^{5/4}}{27w^{4/3}v^{1/2}}$ 64. $\frac{7x^{-3/4}y^{5/2}z^{-2/3}}{56x^{-1/2}y^{1/4}}$

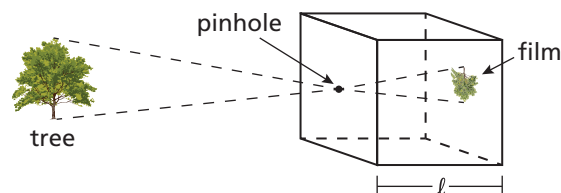
In Exercises 65–70, perform the indicated operation. Assume all variables are positive. (See Example 8.)

65. $12\sqrt[3]{y} + 9\sqrt[3]{y}$
 66. $11\sqrt{2z} - 5\sqrt{2z}$
 67. $3x^{7/2} - 5x^{7/2}$
 68. $7\sqrt[3]{m^7} + 3m^{7/3}$
 69. $\sqrt[4]{16w^{10}} + 2w\sqrt[4]{w^6}$
 70. $(p^{1/2} \cdot p^{1/4}) - \sqrt[4]{16p^3}$

MATHEMATICAL CONNECTIONS In Exercises 71 and 72, find simplified expressions for the perimeter and area of the given figure.



73. **MODELING WITH MATHEMATICS** The optimum diameter d (in millimeters) of the pinhole in a pinhole camera can be modeled by $d = 1.9[(5.5 \times 10^{-4})\ell]^{1/2}$, where ℓ is the length (in millimeters) of the camera box. Find the optimum pinhole diameter for a camera box with a length of 10 centimeters.



74. **MODELING WITH MATHEMATICS** The surface area S (in square centimeters) of a mammal can be modeled by $S = km^{2/3}$, where m is the mass (in grams) of the mammal and k is a constant. The table shows the values of k for different mammals.

Mammal	Rabbit	Human	Bat
Value of k	9.75	11.0	57.5

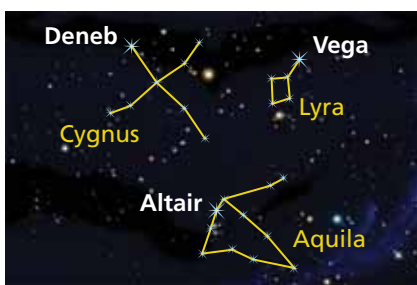
- a. Find the surface area of a bat whose mass is 32 grams.
 b. Find the surface area of a rabbit whose mass is 3.4 kilograms (3.4×10^3 grams).
 c. Find the surface area of a human whose mass is 59 kilograms.

75. **MAKING AN ARGUMENT** Your friend claims it is not possible to simplify the expression $7\sqrt{11} - 9\sqrt{44}$ because it does not contain like radicals. Is your friend correct? Explain your reasoning.

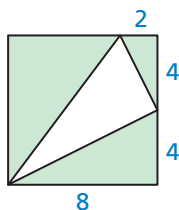
76. **PROBLEM SOLVING** The apparent magnitude of a star is a number that indicates how faint the star is in relation to other stars. The expression $\frac{2.512^{m_1}}{2.512^{m_2}}$ tells how many times fainter a star with apparent magnitude m_1 is than a star with apparent magnitude m_2 .

Star	Apparent magnitude	Constellation
Vega	0.03	Lyra
Altair	0.77	Aquila
Deneb	1.25	Cygnus

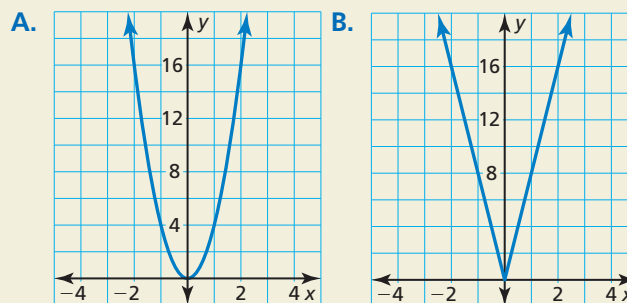
- How many times fainter is Altair than Vega?
- How many times fainter is Deneb than Altair?
- How many times fainter is Deneb than Vega?



77. **CRITICAL THINKING** Find a radical expression for the perimeter of the triangle inscribed in the square shown. Simplify the expression.



78. **HOW DO YOU SEE IT?** Without finding points, match the functions $f(x) = \sqrt{64x^2}$ and $g(x) = \sqrt[3]{64x^6}$ with their graphs. Explain your reasoning.



79. **REWRITING A FORMULA** You have filled two round balloons with water. One balloon contains twice as much water as the other balloon.

- Solve the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, for r .
- Substitute the expression for r from part (a) into the formula for the surface area of a sphere, $S = 4\pi r^2$. Simplify to show that $S = (4\pi)^{1/3}(3V)^{2/3}$.
- Compare the surface areas of the two water balloons using the formula in part (b).

80. **THOUGHT PROVOKING** Determine whether the expressions $(x^2)^{1/6}$ and $(x^{1/6})^2$ are equivalent for all values of x .

81. **DRAWING CONCLUSIONS** Substitute different combinations of odd and even positive integers for m and n in the expression $\sqrt[n]{x^m}$. When you cannot assume x is positive, explain when absolute value is needed in simplifying the expression.

Maintaining Mathematical Proficiency Reviewing what you learned in previous grades and lessons

Identify the focus, directrix, and axis of symmetry of the parabola. Then graph the equation.

(Section 2.3)

82. $y = 2x^2$

83. $y^2 = -x$

84. $y^2 = 4x$

Write a rule for g . Describe the graph of g as a transformation of the graph of f . (Section 4.7)

85. $f(x) = x^4 - 3x^2 - 2x$, $g(x) = -f(x)$

86. $f(x) = x^3 - x$, $g(x) = f(x) - 3$

87. $f(x) = x^3 - 4$, $g(x) = f(x - 2)$

88. $f(x) = x^4 + 2x^3 - 4x^2$, $g(x) = f(2x)$

5.3 Graphing Radical Functions

Essential Question How can you identify the domain and range of a radical function?

EXPLORATION 1 Identifying Graphs of Radical Functions

Work with a partner. Match each function with its graph. Explain your reasoning. Then identify the domain and range of each function.

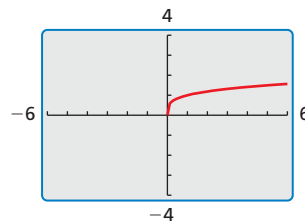
a. $f(x) = \sqrt{x}$

b. $f(x) = \sqrt[3]{x}$

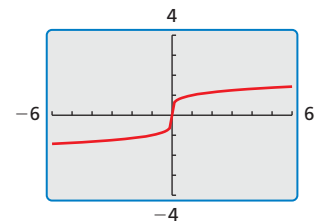
c. $f(x) = \sqrt[4]{x}$

d. $f(x) = \sqrt[5]{x}$

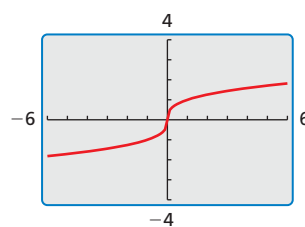
A.



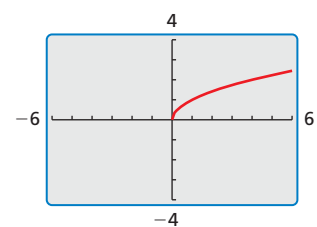
B.



C.



D.



EXPLORATION 2 Identifying Graphs of Transformations

Work with a partner. Match each transformation of $f(x) = \sqrt{x}$ with its graph. Explain your reasoning. Then identify the domain and range of each function.

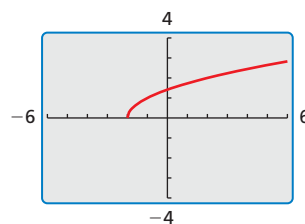
a. $g(x) = \sqrt{x+2}$

b. $g(x) = \sqrt{x-2}$

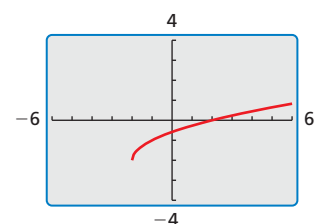
c. $g(x) = \sqrt{x+2} - 2$

d. $g(x) = -\sqrt{x+2}$

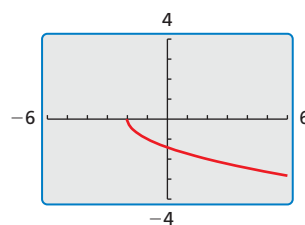
A.



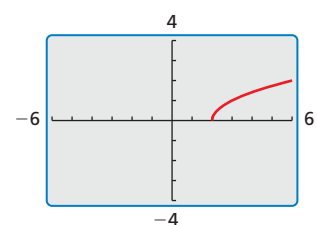
B.



C.



D.



LOOKING FOR STRUCTURE

To be proficient in math, you need to look closely to discern a pattern or structure.

Communicate Your Answer

- How can you identify the domain and range of a radical function?
- Use the results of Exploration 1 to describe how the domain and range of a radical function are related to the index of the radical.

5.3 Lesson

Core Vocabulary

radical function, p. 252

Previous
transformations
parabola
circle

What You Will Learn

- ▶ Graph radical functions.
- ▶ Write transformations of radical functions.
- ▶ Graph parabolas and circles.

Graphing Radical Functions

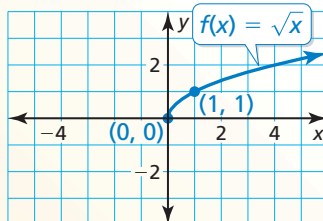
A **radical function** contains a radical expression with the independent variable in the radicand. When the radical is a square root, the function is called a *square root function*. When the radical is a cube root, the function is called a *cube root function*.

Core Concept

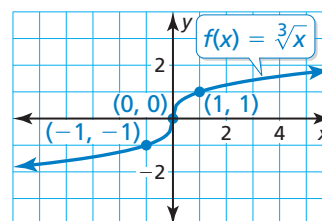
Parent Functions for Square Root and Cube Root Functions

The parent function for the family of square root functions is $f(x) = \sqrt{x}$.

The parent function for the family of cube root functions is $f(x) = \sqrt[3]{x}$.



Domain: $x \geq 0$, Range: $y \geq 0$



Domain and range: All real numbers

STUDY TIP

A *power function* has the form $y = ax^b$, where a is a real number and b is a rational number. Notice that the parent square root function is a power function, where $a = 1$ and $b = \frac{1}{2}$.

EXAMPLE 1 Graphing Radical Functions

Graph each function. Identify the domain and range of each function.

a. $f(x) = \sqrt{\frac{1}{4}x}$

b. $g(x) = -3\sqrt[3]{x}$

SOLUTION

a. Make a table of values and sketch the graph.

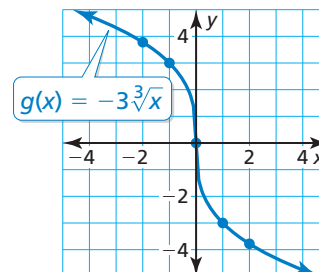
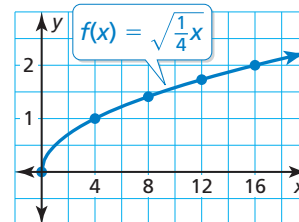
x	0	4	8	12	16
y	0	1	1.41	1.73	2

- ▶ The radicand of a square root must be nonnegative. So, the domain is $x \geq 0$. The range is $y \geq 0$.

b. Make a table of values and sketch the graph.

x	-2	-1	0	1	2
y	3.78	3	0	-3	-3.78

- ▶ The radicand of a cube root can be any real number. So, the domain and range are all real numbers.



LOOKING FOR STRUCTURE

Example 1(a) uses x -values that are multiples of 4 so that the radicand is an integer.

In Example 1, notice that the graph of f is a horizontal stretch of the graph of the parent square root function. The graph of g is a vertical stretch and a reflection in the x -axis of the graph of the parent cube root function. You can transform graphs of radical functions in the same way you transformed graphs of functions previously.

Core Concept

Transformation	$f(x)$ Notation	Examples
Horizontal Translation Graph shifts left or right.	$f(x - h)$	$g(x) = \sqrt{x - 2}$ 2 units right $g(x) = \sqrt{x + 3}$ 3 units left
Vertical Translation Graph shifts up or down.	$f(x) + k$	$g(x) = \sqrt{x} + 7$ 7 units up $g(x) = \sqrt{x} - 1$ 1 unit down
Reflection Graph flips over x - or y -axis.	$f(-x)$ $-f(x)$	$g(x) = \sqrt{-x}$ in the y -axis $g(x) = -\sqrt{x}$ in the x -axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y -axis.	$f(ax)$	$g(x) = \sqrt{3x}$ shrink by a factor of $\frac{1}{3}$ $g(x) = \sqrt{\frac{1}{2}x}$ stretch by a factor of 2
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x -axis.	$a \cdot f(x)$	$g(x) = 4\sqrt{x}$ stretch by a factor of 4 $g(x) = \frac{1}{5}\sqrt{x}$ shrink by a factor of $\frac{1}{5}$

EXAMPLE 2 Transforming Radical Functions

Describe the transformation of f represented by g . Then graph each function.

a. $f(x) = \sqrt{x}$, $g(x) = \sqrt{x - 3} + 4$

b. $f(x) = \sqrt[3]{x}$, $g(x) = \sqrt[3]{-8x}$

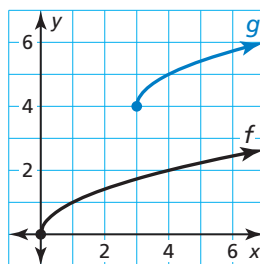
LOOKING FOR STRUCTURE

In Example 2(b), you can use the Product Property of Radicals to write $g(x) = -2\sqrt[3]{x}$. So, you can also describe the graph of g as a vertical stretch by a factor of 2 and a reflection in the x -axis of the graph of f .

SOLUTION

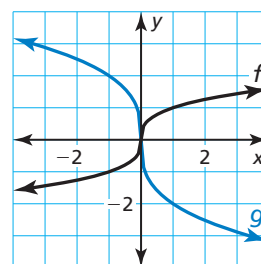
a. Notice that the function is of the form $g(x) = \sqrt{x - h} + k$, where $h = 3$ and $k = 4$.

► So, the graph of g is a translation 3 units right and 4 units up of the graph of f .



b. Notice that the function is of the form $g(x) = \sqrt[3]{ax}$, where $a = -8$.

► So, the graph of g is a horizontal shrink by a factor of $\frac{1}{8}$ and a reflection in the y -axis of the graph of f .



Monitoring Progress



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- Graph $g(x) = \sqrt{x + 1}$. Identify the domain and range of the function.
- Describe the transformation of $f(x) = \sqrt[3]{x}$ represented by $g(x) = -\sqrt[3]{x} - 2$. Then graph each function.

Writing Transformations of Radical Functions

EXAMPLE 3 Modeling with Mathematics



Self-Portrait of
NASA's Mars Rover Curiosity

The function $E(d) = 0.25\sqrt{d}$ approximates the number of seconds it takes a dropped object to fall d feet on Earth. The function $M(d) = 1.6 \cdot E(d)$ approximates the number of seconds it takes a dropped object to fall d feet on Mars. Write a rule for M . How long does it take a dropped object to fall 64 feet on Mars?

SOLUTION

1. Understand the Problem You are given a function that represents the number of seconds it takes a dropped object to fall d feet on Earth. You are asked to write a similar function for Mars and then evaluate the function for a given input.

2. Make a Plan Multiply $E(d)$ by 1.6 to write a rule for M . Then find $M(64)$.

3. Solve the Problem

$$M(d) = 1.6 \cdot E(d)$$

$$= 1.6 \cdot 0.25\sqrt{d} \quad \text{Substitute } 0.25\sqrt{d} \text{ for } E(d).$$

$$= 0.4\sqrt{d} \quad \text{Simplify.}$$

Next, find $M(64)$.

$$M(64) = 0.4\sqrt{64} = 0.4(8) = 3.2$$

► It takes a dropped object about 3.2 seconds to fall 64 feet on Mars.

4. Look Back Use the original functions to check your solution.

$$E(64) = 0.25\sqrt{64} = 2 \quad M(64) = 1.6 \cdot E(64) = 1.6 \cdot 2 = 3.2 \quad \checkmark$$

EXAMPLE 4 Writing a Transformed Radical Function

Let the graph of g be a horizontal shrink by a factor of $\frac{1}{6}$ followed by a translation 3 units to the left of the graph of $f(x) = \sqrt[3]{x}$. Write a rule for g .

SOLUTION

Step 1 First write a function h that represents the horizontal shrink of f .

$$h(x) = f(6x) \quad \text{Multiply the input by } 1 \div \frac{1}{6} = 6.$$

$$= \sqrt[3]{6x} \quad \text{Replace } x \text{ with } 6x \text{ in } f(x).$$

Step 2 Then write a function g that represents the translation of h .

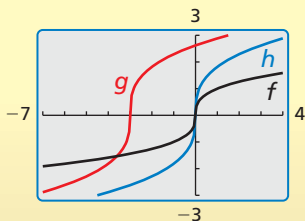
$$g(x) = h(x + 3) \quad \text{Subtract } -3, \text{ or add } 3, \text{ to the input.}$$

$$= \sqrt[3]{6(x + 3)} \quad \text{Replace } x \text{ with } x + 3 \text{ in } h(x).$$

$$= \sqrt[3]{6x + 18} \quad \text{Distributive Property}$$

► The transformed function is $g(x) = \sqrt[3]{6x + 18}$.

Check



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- WHAT IF?** In Example 3, the function $N(d) = 2.4 \cdot E(d)$ approximates the number of seconds it takes a dropped object to fall d feet on the Moon. Write a rule for N . How long does it take a dropped object to fall 25 feet on the Moon?
- In Example 4, is the transformed function the same when you perform the translation followed by the horizontal shrink? Explain your reasoning.

Graphing Parabolas and Circles

To graph parabolas and circles using a graphing calculator, first solve their equations for y to obtain radical functions. Then graph the functions.

EXAMPLE 5 Graphing a Parabola (Horizontal Axis of Symmetry)

Use a graphing calculator to graph $\frac{1}{2}y^2 = x$. Identify the vertex and the direction that the parabola opens.

SOLUTION

Step 1 Solve for y .

$$\frac{1}{2}y^2 = x$$

$$y^2 = 2x$$

$$y = \pm\sqrt{2x}$$

Write the original equation.

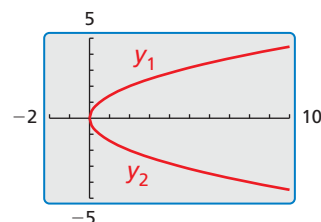
Multiply each side by 2.

Take square root of each side.

Step 2 Graph both radical functions.

$$y_1 = \sqrt{2x}$$

$$y_2 = -\sqrt{2x}$$



STUDY TIP

Notice y_1 is a function and y_2 is a function, but $\frac{1}{2}y^2 = x$ is not a function.

► The vertex is $(0, 0)$ and the parabola opens right.

EXAMPLE 6 Graphing a Circle (Center at the Origin)

Use a graphing calculator to graph $x^2 + y^2 = 16$. Identify the radius and the intercepts.

SOLUTION

Step 1 Solve for y .

$$x^2 + y^2 = 16$$

$$y^2 = 16 - x^2$$

$$y = \pm\sqrt{16 - x^2}$$

Write the original equation.

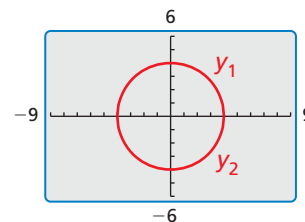
Subtract x^2 from each side.

Take square root of each side.

Step 2 Graph both radical functions using a square viewing window.

$$y_1 = \sqrt{16 - x^2}$$

$$y_2 = -\sqrt{16 - x^2}$$



► The radius is 4 units. The x -intercepts are ± 4 . The y -intercepts are also ± 4 .

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- Use a graphing calculator to graph $-4y^2 = x + 1$. Identify the vertex and the direction that the parabola opens.
- Use a graphing calculator to graph $x^2 + y^2 = 25$. Identify the radius and the intercepts.

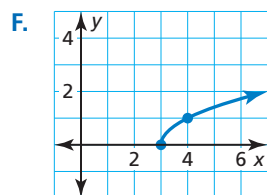
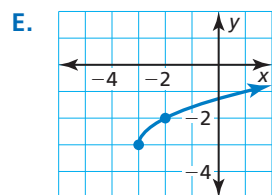
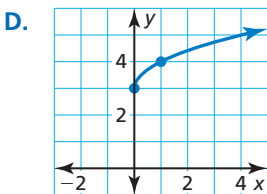
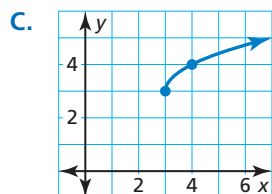
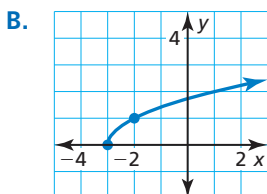
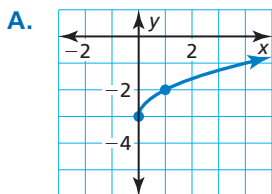
Vocabulary and Core Concept Check

- COMPLETE THE SENTENCE** Square root functions and cube root functions are examples of _____ functions.
- COMPLETE THE SENTENCE** When graphing $y = a\sqrt[3]{x-h} + k$, translate the graph of $y = a\sqrt[3]{x}$ h units _____ and k units _____.

Monitoring Progress and Modeling with Mathematics

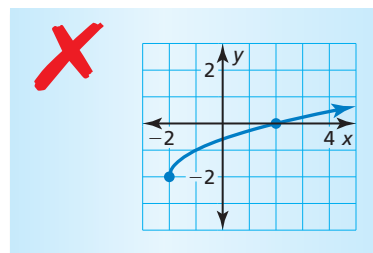
In Exercises 3–8, match the function with its graph.

- | | |
|----------------------------|----------------------------|
| 3. $f(x) = \sqrt{x+3}$ | 4. $h(x) = \sqrt{x} + 3$ |
| 5. $f(x) = \sqrt{x-3}$ | 6. $g(x) = \sqrt{x} - 3$ |
| 7. $h(x) = \sqrt{x+3} - 3$ | 8. $f(x) = \sqrt{x-3} + 3$ |



In Exercises 19–26, describe the transformation of f represented by g . Then graph each function. (See Example 2.)


- $f(x) = \sqrt{x}$, $g(x) = \sqrt{x+1} + 8$
- $f(x) = \sqrt{x}$, $g(x) = 2\sqrt{x-1}$
- $f(x) = \sqrt[3]{x}$, $g(x) = -\sqrt[3]{x} - 1$
- $f(x) = \sqrt[3]{x}$, $g(x) = \sqrt[3]{x+4} - 5$
- $f(x) = x^{1/2}$, $g(x) = \frac{1}{4}(-x)^{1/2}$
- $f(x) = x^{1/3}$, $g(x) = \frac{1}{3}x^{1/3} + 6$
- $f(x) = \sqrt[4]{x}$, $g(x) = 2\sqrt[4]{x+5} - 4$
- $f(x) = \sqrt[5]{x}$, $g(x) = \sqrt[5]{-32x} + 3$
- ERROR ANALYSIS** Describe and correct the error in graphing $f(x) = \sqrt{x-2} - 2$.



In Exercises 9–18, graph the function. Identify the domain and range of the function. (See Example 1.)

- | | |
|------------------------------------|---------------------------------------|
| 9. $h(x) = \sqrt{x} + 4$ | 10. $g(x) = \sqrt{x} - 5$ |
| 11. $g(x) = -\sqrt[3]{2x}$ | 12. $f(x) = \sqrt[3]{-5x}$ |
| 13. $g(x) = \frac{1}{5}\sqrt{x-3}$ | 14. $f(x) = \frac{1}{2}\sqrt[3]{x+6}$ |
| 15. $f(x) = (6x)^{1/2} + 3$ | 16. $g(x) = -3(x+1)^{1/3}$ |
| 17. $h(x) = -\sqrt[4]{x}$ | 18. $h(x) = \sqrt[5]{2x}$ |

- ERROR ANALYSIS** Describe and correct the error in describing the transformation of the parent square root function represented by $g(x) = \sqrt{\frac{1}{2}x} + 3$.

 The graph of g is a horizontal shrink by a factor of $\frac{1}{2}$ and a translation 3 units up of the parent square root function.

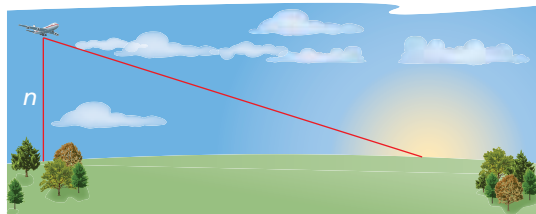
USING TOOLS In Exercises 29–34, use a graphing calculator to graph the function. Then identify the domain and range of the function.

29. $g(x) = \sqrt{x^2 + x}$ 30. $h(x) = \sqrt{x^2 - 2x}$
 31. $f(x) = \sqrt[3]{x^2 + x}$ 32. $f(x) = \sqrt[3]{3x^2 - x}$
 33. $f(x) = \sqrt{2x^2 + x + 1}$ 34. $h(x) = \sqrt{\frac{1}{2}x^2 - 3x + 4}$

ABSTRACT REASONING In Exercises 35–38, complete the statement with *sometimes*, *always*, or *never*.

35. The domain of the function $y = a\sqrt{x}$ is _____ $x \geq 0$.
 36. The range of the function $y = a\sqrt{x}$ is _____ $y \geq 0$.
 37. The domain and range of the function $y = \sqrt[3]{x - h} + k$ are _____ all real numbers.
 38. The domain of the function $y = a\sqrt{-x} + k$ is _____ $x \geq 0$.

39. **PROBLEM SOLVING** The distance (in miles) a pilot can see to the horizon can be approximated by $E(n) = 1.22\sqrt{n}$, where n is the plane's altitude (in feet above sea level) on Earth. The function $M(n) = 0.75E(n)$ approximates the distance a pilot can see to the horizon n feet above the surface of Mars. Write a rule for M . What is the distance a pilot can see to the horizon from an altitude of 10,000 feet above Mars? (See Example 3.)



40. **MODELING WITH MATHEMATICS** The speed (in knots) of sound waves in air can be modeled by

$$v(K) = 643.855\sqrt{\frac{K}{273.15}}$$

where K is the air temperature (in kelvin). The speed (in meters per second) of sound waves in air can be modeled by

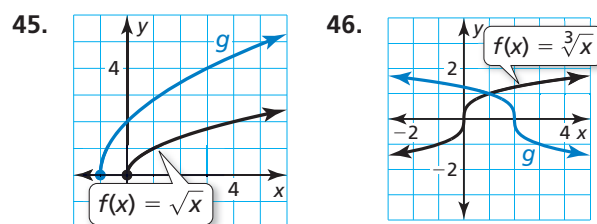
$$s(K) = \frac{v(K)}{1.944}$$

Write a rule for s . What is the speed (in meters per second) of sound waves when the air temperature is 305 kelvin?

In Exercises 41–44, write a rule for g described by the transformations of the graph of f . (See Example 4.)

41. Let g be a vertical stretch by a factor of 2, followed by a translation 2 units up of the graph of $f(x) = \sqrt{x} + 3$.
 42. Let g be a reflection in the y -axis, followed by a translation 1 unit right of the graph of $f(x) = 2\sqrt[3]{x - 1}$.
 43. Let g be a horizontal shrink by a factor of $\frac{2}{3}$, followed by a translation 4 units left of the graph of $f(x) = \sqrt{6x}$.
 44. Let g be a translation 1 unit down and 5 units right, followed by a reflection in the x -axis of the graph of $f(x) = -\frac{1}{2}\sqrt[4]{x} + \frac{3}{2}$.

In Exercises 45 and 46, write a rule for g .



In Exercises 47–50, write a rule for g that represents the indicated transformation of the graph of f .

47. $f(x) = 2\sqrt{x}$, $g(x) = f(x + 3)$
 48. $f(x) = \frac{1}{3}\sqrt{x - 1}$, $g(x) = -f(x) + 9$
 49. $f(x) = -\sqrt{x^2 - 2}$, $g(x) = -2f(x + 5)$
 50. $f(x) = \sqrt[3]{x^2 + 10x}$, $g(x) = \frac{1}{4}f(-x) + 6$

In Exercises 51–56, use a graphing calculator to graph the equation of the parabola. Identify the vertex and the direction that the parabola opens. (See Example 5.)

51. $\frac{1}{4}y^2 = x$ 52. $3y^2 = x$
 53. $-8y^2 + 2 = x$ 54. $2y^2 = x - 4$
 55. $x + 8 = \frac{1}{5}y^2$ 56. $\frac{1}{2}x = y^2 - 4$

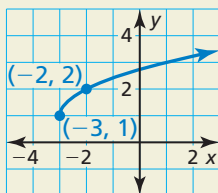
In Exercises 57–62, use a graphing calculator to graph the equation of the circle. Identify the radius and the intercepts. (See Example 6.)

57. $x^2 + y^2 = 9$ 58. $x^2 + y^2 = 4$
 59. $1 - y^2 = x^2$ 60. $64 - x^2 = y^2$
 61. $-y^2 = x^2 - 36$ 62. $x^2 = 100 - y^2$

63. **MODELING WITH MATHEMATICS** The *period* of a pendulum is the time the pendulum takes to complete one back-and-forth swing. The period T (in seconds) can be modeled by the function $T = 1.11\sqrt{\ell}$, where ℓ is the length (in feet) of the pendulum. Graph the function. Estimate the length of a pendulum with a period of 2 seconds. Explain your reasoning.



64. **HOW DO YOU SEE IT?** Does the graph represent a square root function or a cube root function? Explain. What are the domain and range of the function?



65. **PROBLEM SOLVING** For a drag race car with a total weight of 3500 pounds, the speed s (in miles per hour) at the end of a race can be modeled by $s = 14.8\sqrt[3]{p}$, where p is the power (in horsepower). Graph the function.
- Determine the power of a 3500-pound car that reaches a speed of 200 miles per hour.
 - What is the average rate of change in speed as the power changes from 1000 horsepower to 1500 horsepower?

66. **THOUGHT PROVOKING** The graph of a radical function f passes through the points $(3, 1)$ and $(4, 0)$. Write two different functions that could represent $f(x + 2) + 1$. Explain.

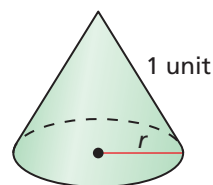
67. **MULTIPLE REPRESENTATIONS** The terminal velocity v_t (in feet per second) of a skydiver who weighs 140 pounds is given by

$$v_t = 33.7\sqrt{\frac{140}{A}}$$

where A is the cross-sectional surface area (in square feet) of the skydiver. The table shows the terminal velocities (in feet per second) for various surface areas (in square feet) of a skydiver who weighs 165 pounds.

Cross-sectional surface area, A	Terminal velocity, v_t
1	432.9
3	249.9
5	193.6
7	163.6

- Which skydiver has a greater terminal velocity for each value of A given in the table?
 - Describe how the different values of A given in the table relate to the possible positions of the falling skydiver.
68. **MATHEMATICAL CONNECTIONS** The surface area S of a right circular cone with a slant height of 1 unit is given by $S = \pi r + \pi r^2$, where r is the radius of the cone.



- Use completing the square to show that

$$r = \frac{1}{\sqrt{\pi}}\sqrt{S + \frac{\pi}{4}} - \frac{1}{2}.$$
- Graph the equation in part (a) using a graphing calculator. Then find the radius of a right circular cone with a slant height of 1 unit and a surface area of $\frac{3\pi}{4}$ square units.

Maintaining Mathematical Proficiency Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solutions. *(Skills Review Handbook)*

69. $|3x + 2| = 5$ 70. $|4x + 9| = -7$ 71. $|2x - 6| = |x|$ 72. $|x + 8| = |2x + 2|$

Solve the inequality. *(Section 3.6)*

73. $x^2 + 7x + 12 < 0$ 74. $x^2 - 10x + 25 \geq 4$ 75. $2x^2 + 6 > 13x$ 76. $\frac{1}{8}x^2 + x \leq -2$

5.1–5.3 What Did You Learn?

Core Vocabulary

n th root of a , *p.* 238

index of a radical, *p.* 238

simplest form of a radical, *p.* 245

conjugate, *p.* 246

like radicals, *p.* 246

radical function, *p.* 252

Core Concepts

Section 5.1

Real n th Roots of a , *p.* 238

Rational Exponents, *p.* 239

Section 5.2

Properties of Rational Exponents, *p.* 244

Properties of Radicals, *p.* 245

Section 5.3

Parent Functions for Square Root and Cube Root Functions, *p.* 252

Transformations of Radical Functions, *p.* 253

Mathematical Practices

1. How can you use definitions to explain your reasoning in Exercises 21–24 on page 241?
2. How did you use structure to solve Exercise 76 on page 250?
3. How can you check that your answer is reasonable in Exercise 39 on page 257?
4. How can you make sense of the terms of the surface area formula given in Exercise 68 on page 258?

Study Skills

Analyzing Your Errors

Application Errors

What Happens: You can do numerical problems, but you struggle with problems that have context.

How to Avoid This Error: Do not just mimic the steps of solving an application problem. Explain out loud what the question is asking and why you are doing each step. After solving the problem, ask yourself, “Does my solution make sense?”



5.1–5.3 Quiz

Find the indicated real n th root(s) of a . (Section 5.1)

- $n = 4, a = 81$
- $n = 5, a = -1024$
- Evaluate (a) $16^{3/4}$ and (b) $125^{2/3}$ without using a calculator. Explain your reasoning. (Section 5.1)

Find the real solution(s) of the equation. Round your answer to two decimal places. (Section 5.1)

- $2x^6 = 1458$
- $(x + 6)^3 = 28$

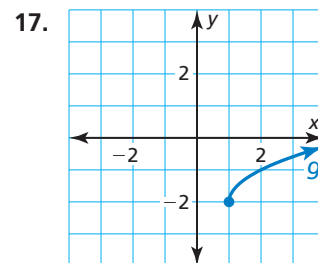
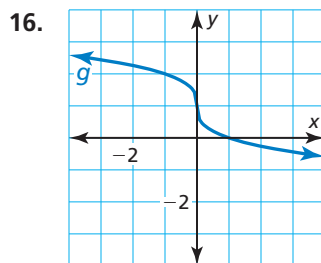
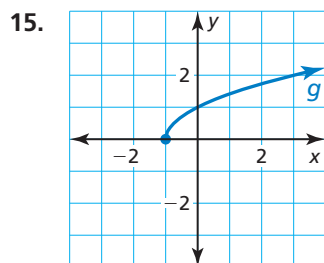
Simplify the expression. (Section 5.2)

- $\left(\frac{48^{1/4}}{6^{1/4}}\right)^6$
- $\sqrt[4]{3} \cdot \sqrt[4]{432}$
- $\frac{1}{3 + \sqrt{2}}$
- $\sqrt[3]{16} - 5\sqrt[3]{2}$
- Simplify $\sqrt[8]{x^9y^8z^{16}}$. (Section 5.2)

Write the expression in simplest form. Assume all variables are positive. (Section 5.2)

- $\sqrt[3]{216p^9}$
- $\frac{\sqrt[5]{32}}{\sqrt[5]{m^3}}$
- $\sqrt[4]{n^4q} + 7n\sqrt[4]{q}$
- Graph $f(x) = 2\sqrt[3]{x} + 1$. Identify the domain and range of the function. (Section 5.3)

Describe the transformation of the parent function represented by the graph of g . Then write a rule for g . (Section 5.3)



- Use a graphing calculator to graph $x = 3y^2 - 6$. Identify the vertex and direction the parabola opens. (Section 5.3)
- A jeweler is setting a stone cut in the shape of a regular octahedron. A regular octahedron is a solid with eight equilateral triangles as faces, as shown. The formula for the volume of the stone is $V = 0.47s^3$, where s is the side length (in millimeters) of an edge of the stone. The volume of the stone is 161 cubic millimeters. Find the length of an edge of the stone. (Section 5.1)
- An investigator can determine how fast a car was traveling just prior to an accident using the model $s = 4\sqrt{d}$, where s is the speed (in miles per hour) of the car and d is the length (in feet) of the skid marks. Graph the model. The length of the skid marks of a car is 90 feet. Was the car traveling at the posted speed limit prior to the accident? Explain your reasoning. (Section 5.3)



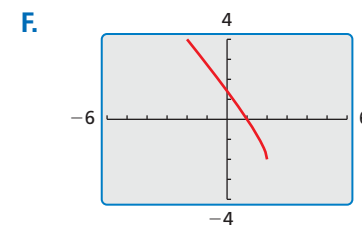
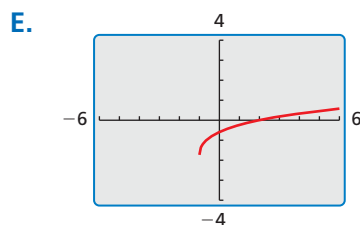
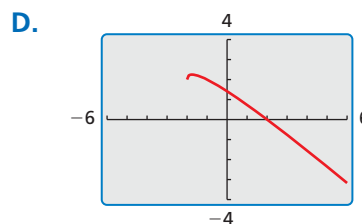
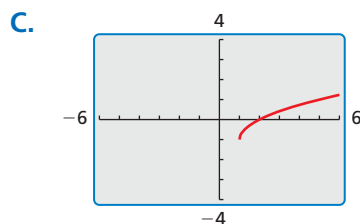
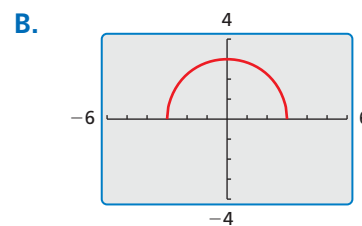
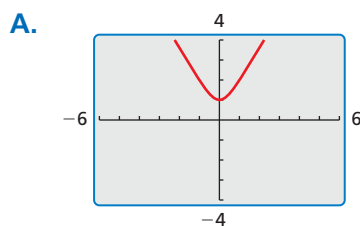
5.4 Solving Radical Equations and Inequalities

Essential Question How can you solve a radical equation?

EXPLORATION 1 Solving Radical Equations

Work with a partner. Match each radical equation with the graph of its related radical function. Explain your reasoning. Then use the graph to solve the equation, if possible. Check your solutions.

- a. $\sqrt{x-1} - 1 = 0$ b. $\sqrt{2x+2} - \sqrt{x+4} = 0$ c. $\sqrt{9-x^2} = 0$
 d. $\sqrt{x+2} - x = 0$ e. $\sqrt{-x+2} - x = 0$ f. $\sqrt{3x^2+1} = 0$



LOOKING FOR STRUCTURE

To be proficient in math, you need to look closely to discern a pattern or structure.

EXPLORATION 2 Solving Radical Equations

Work with a partner. Look back at the radical equations in Exploration 1. Suppose that you did not know how to solve the equations using a graphical approach.

- a. Show how you could use a *numerical approach* to solve one of the equations. For instance, you might use a spreadsheet to create a table of values.
 b. Show how you could use an *analytical approach* to solve one of the equations. For instance, look at the similarities between the equations in Exploration 1. What first step may be necessary so you could square each side to eliminate the radical(s)? How would you proceed to find the solution?

Communicate Your Answer

3. How can you solve a radical equation?
 4. Would you prefer to use a graphical, numerical, or analytical approach to solve the given equation? Explain your reasoning. Then solve the equation.

$$\sqrt{x+3} - \sqrt{x-2} = 1$$

5.4 Lesson

Core Vocabulary

radical equation, p. 262
extraneous solutions, p. 263

Previous

rational exponents
radical expressions
solving quadratic equations

What You Will Learn

- ▶ Solve equations containing radicals and rational exponents.
- ▶ Solve radical inequalities.

Solving Equations

Equations with radicals that have variables in their radicands are called **radical equations**. An example of a radical equation is $2\sqrt{x+1} = 4$.

Core Concept

Solving Radical Equations

To solve a radical equation, follow these steps:

- Step 1** Isolate the radical on one side of the equation, if necessary.
- Step 2** Raise each side of the equation to the same exponent to eliminate the radical and obtain a linear, quadratic, or other polynomial equation.
- Step 3** Solve the resulting equation using techniques you learned in previous chapters. Check your solution.

EXAMPLE 1 Solving Radical Equations

Solve (a) $2\sqrt{x+1} = 4$ and (b) $\sqrt[3]{2x-9} - 1 = 2$.

SOLUTION

a. $2\sqrt{x+1} = 4$

Write the original equation.

$$\sqrt{x+1} = 2$$

Divide each side by 2.

$$(\sqrt{x+1})^2 = 2^2$$

Square each side to eliminate the radical.

$$x+1 = 4$$

Simplify.

$$x = 3$$

Subtract 1 from each side.

▶ The solution is $x = 3$.

b. $\sqrt[3]{2x-9} - 1 = 2$

Write the original equation.

$$\sqrt[3]{2x-9} = 3$$

Add 1 to each side.

$$(\sqrt[3]{2x-9})^3 = 3^3$$

Cube each side to eliminate the radical.

$$2x-9 = 27$$

Simplify.

$$2x = 36$$

Add 9 to each side.

$$x = 18$$

Divide each side by 2.

▶ The solution is $x = 18$.

Check

$$2\sqrt{3+1} \stackrel{?}{=} 4$$

$$2\sqrt{4} \stackrel{?}{=} 4$$

$$4 = 4 \quad \checkmark$$

Check

$$\sqrt[3]{2(18)-9} - 1 \stackrel{?}{=} 2$$

$$\sqrt[3]{27} - 1 \stackrel{?}{=} 2$$

$$2 = 2 \quad \checkmark$$

Monitoring Progress



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Solve the equation. Check your solution.

1. $\sqrt[3]{x} - 9 = -6$

2. $\sqrt{x+25} = 2$

3. $2\sqrt[3]{x-3} = 4$



EXAMPLE 2 Solving a Real-Life Problem

In a hurricane, the mean sustained wind velocity v (in meters per second) can be modeled by $v(p) = 6.3\sqrt{1013 - p}$, where p is the air pressure (in millibars) at the center of the hurricane. Estimate the air pressure at the center of the hurricane when the mean sustained wind velocity is 54.5 meters per second.

SOLUTION

$$v(p) = 6.3\sqrt{1013 - p}$$

Write the original function.

$$54.5 = 6.3\sqrt{1013 - p}$$

Substitute 54.5 for $v(p)$.

$$8.65 \approx \sqrt{1013 - p}$$

Divide each side by 6.3.

$$8.65^2 \approx (\sqrt{1013 - p})^2$$

Square each side.

$$74.8 \approx 1013 - p$$

Simplify.

$$-938.2 \approx -p$$

Subtract 1013 from each side.

$$938.2 \approx p$$

Divide each side by -1 .

► The air pressure at the center of the hurricane is about 938 millibars.

ATTEND TO PRECISION

To understand how extraneous solutions can be introduced, consider the equation $\sqrt{x} = -3$. This equation has no real solution; however, you obtain $x = 9$ after squaring each side.

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4. **WHAT IF?** Estimate the air pressure at the center of the hurricane when the mean sustained wind velocity is 48.3 meters per second.

Raising each side of an equation to the same exponent may introduce solutions that are *not* solutions of the original equation. These solutions are called **extraneous solutions**. When you use this procedure, you should always check each apparent solution in the *original* equation.

EXAMPLE 3 Solving an Equation with an Extraneous Solution

Solve $x + 1 = \sqrt{7x + 15}$.

SOLUTION

$$x + 1 = \sqrt{7x + 15}$$

Write the original equation.

$$(x + 1)^2 = (\sqrt{7x + 15})^2$$

Square each side.

$$x^2 + 2x + 1 = 7x + 15$$

Expand left side and simplify right side.

$$x^2 - 5x - 14 = 0$$

Write in standard form.

$$(x - 7)(x + 2) = 0$$

Factor.

$$x - 7 = 0 \quad \text{or} \quad x + 2 = 0$$

Zero-Product Property

$$x = 7 \quad \text{or} \quad x = -2$$

Solve for x .

Check	$7 + 1 \stackrel{?}{=} \sqrt{7(7) + 15}$	$-2 + 1 \stackrel{?}{=} \sqrt{7(-2) + 15}$
	$8 \stackrel{?}{=} \sqrt{64}$	$-1 \stackrel{?}{=} \sqrt{1}$
	$8 = 8$ ✓	$-1 \neq 1$ ✗

► The apparent solution $x = -2$ is extraneous. So, the only solution is $x = 7$.

EXAMPLE 4 Solving an Equation with Two Radicals

Solve $\sqrt{x+2} + 1 = \sqrt{3-x}$.

SOLUTION

$$\sqrt{x+2} + 1 = \sqrt{3-x}$$

$$(\sqrt{x+2} + 1)^2 = (\sqrt{3-x})^2$$

$$x + 2 + 2\sqrt{x+2} + 1 = 3 - x$$

$$2\sqrt{x+2} = -2x$$

$$\sqrt{x+2} = -x$$

$$(\sqrt{x+2})^2 = (-x)^2$$

$$x + 2 = x^2$$

$$0 = x^2 - x - 2$$

$$0 = (x - 2)(x + 1)$$

$$x - 2 = 0 \quad \text{or} \quad x + 1 = 0$$

$$x = 2 \quad \text{or} \quad x = -1$$

Write the original equation.

Square each side.

Expand left side and simplify right side.

Isolate radical expression.

Divide each side by 2.

Square each side.

Simplify.

Write in standard form.

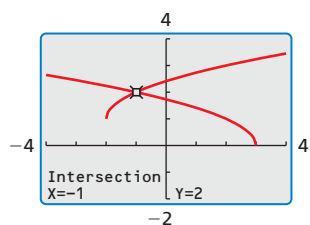
Factor.

Zero-Product Property

Solve for x .

ANOTHER METHOD

You can also graph each side of the equation and find the x -value where the graphs intersect.



Check

$\sqrt{2+2} + 1 \stackrel{?}{=} \sqrt{3-2}$	$\sqrt{-1+2} + 1 \stackrel{?}{=} \sqrt{3-(-1)}$
$\sqrt{4} + 1 \stackrel{?}{=} \sqrt{1}$	$\sqrt{1} + 1 \stackrel{?}{=} \sqrt{4}$
$3 \neq 1$ ❌	$2 = 2$ ✅

▶ The apparent solution $x = 2$ is extraneous. So, the only solution is $x = -1$.

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Solve the equation. Check your solution(s).

5. $\sqrt{10x+9} = x+3$ 6. $\sqrt{2x+5} = \sqrt{x+7}$ 7. $\sqrt{x+6} - 2 = \sqrt{x-2}$

When an equation contains a power with a rational exponent, you can solve the equation using a procedure similar to the one for solving radical equations. In this case, you first isolate the power and then raise each side of the equation to the reciprocal of the rational exponent.

EXAMPLE 5 Solving an Equation with a Rational Exponent

Solve $(2x)^{3/4} + 2 = 10$.

SOLUTION

$$(2x)^{3/4} + 2 = 10 \quad \text{Write the original equation.}$$

$$(2x)^{3/4} = 8 \quad \text{Subtract 2 from each side.}$$

$$[(2x)^{3/4}]^{4/3} = 8^{4/3} \quad \text{Raise each side to the four-thirds.}$$

$$2x = 16 \quad \text{Simplify.}$$

$$x = 8 \quad \text{Divide each side by 2.}$$

Check

$$(2 \cdot 8)^{3/4} + 2 \stackrel{?}{=} 10$$

$$16^{3/4} + 2 \stackrel{?}{=} 10$$

$$10 = 10 \quad \checkmark$$

▶ The solution is $x = 8$.

EXAMPLE 6**Solving an Equation with a Rational Exponent**Solve $(x + 30)^{1/2} = x$.**SOLUTION**

$$(x + 30)^{1/2} = x$$

$$[(x + 30)^{1/2}]^2 = x^2$$

$$x + 30 = x^2$$

$$0 = x^2 - x - 30$$

$$0 = (x - 6)(x + 5)$$

$$x - 6 = 0 \quad \text{or} \quad x + 5 = 0$$

$$x = 6 \quad \text{or} \quad x = -5$$

Write the original equation.

Square each side.

Simplify.

Write in standard form.

Factor.

Zero-Product Property

Solve for x .**Check**

$$(6 + 30)^{1/2} \stackrel{?}{=} 6$$

$$36^{1/2} \stackrel{?}{=} 6$$

$$6 = 6 \quad \checkmark$$

$$(-5 + 30)^{1/2} \stackrel{?}{=} -5$$

$$25^{1/2} \stackrel{?}{=} -5$$

$$5 \neq -5 \quad \times$$

► The apparent solution $x = -5$ is extraneous. So, the only solution is $x = 6$.**Monitoring Progress**Help in English and Spanish at BigIdeasMath.com

Solve the equation. Check your solution(s).

8. $(3x)^{1/3} = -3$

9. $(x + 6)^{1/2} = x$

10. $(x + 2)^{3/4} = 8$

Solving Radical InequalitiesTo solve a simple radical inequality of the form $\sqrt[n]{u} < d$, where u is an algebraic expression and d is a nonnegative number, raise each side to the exponent n . This procedure also works for $>$, \leq , and \geq . Be sure to consider the possible values of the radicand.**EXAMPLE 7****Solving a Radical Inequality**Solve $3\sqrt{x - 1} \leq 12$.**SOLUTION****Step 1** Solve for x .

$$3\sqrt{x - 1} \leq 12$$

$$\sqrt{x - 1} \leq 4$$

$$x - 1 \leq 16$$

$$x \leq 17$$

Write the original inequality.

Divide each side by 3.

Square each side.

Add 1 to each side.

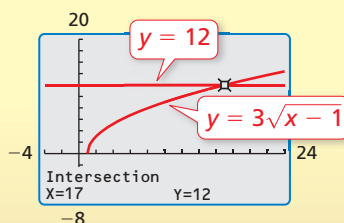
Step 2 Consider the radicand.

$$x - 1 \geq 0$$

$$x \geq 1$$

The radicand cannot be negative.

Add 1 to each side.

► So, the solution is $1 \leq x \leq 17$.**Check****Monitoring Progress**Help in English and Spanish at BigIdeasMath.com11. Solve (a) $2\sqrt{x} - 3 \geq 3$ and (b) $4\sqrt[3]{x + 1} < 8$.

Vocabulary and Core Concept Check

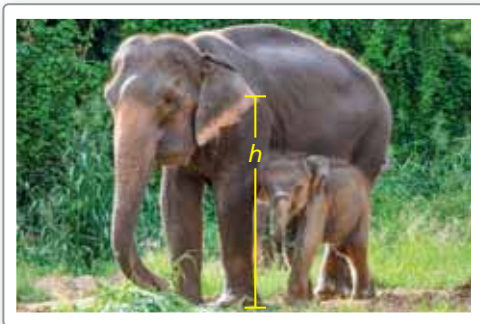
- VOCABULARY** Is the equation $3x - \sqrt{2} = \sqrt{6}$ a radical equation? Explain your reasoning.
- WRITING** Explain the steps you should use to solve $\sqrt{x} + 10 < 15$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–12, solve the equation. Check your solution. (See Example 1.)

- $\sqrt{5x+1} = 6$
- $\sqrt{3x+10} = 8$
- $\sqrt[3]{x-16} = 2$
- $\sqrt[3]{x} - 10 = -7$
- $-2\sqrt{24x} + 13 = -11$
- $8\sqrt[3]{10x} - 15 = 17$
- $\frac{1}{5}\sqrt[3]{3x} + 10 = 8$
- $\sqrt{2x} - \frac{2}{3} = 0$
- $2\sqrt[5]{x} + 7 = 15$
- $\sqrt[4]{4x} - 13 = -15$

- 13. MODELING WITH MATHEMATICS** Biologists have discovered that the shoulder height h (in centimeters) of a male Asian elephant can be modeled by $h = 62.5\sqrt[3]{t} + 75.8$, where t is the age (in years) of the elephant. Determine the age of an elephant with a shoulder height of 250 centimeters. (See Example 2.)



- 14. MODELING WITH MATHEMATICS** In an amusement park ride, a rider suspended by cables swings back and forth from a tower. The maximum speed v (in meters per second) of the rider can be approximated by $v = \sqrt{2gh}$, where h is the height (in meters) at the top of each swing and g is the acceleration due to gravity ($g \approx 9.8$ m/sec²). Determine the height at the top of the swing of a rider whose maximum speed is 15 meters per second.

In Exercises 15–26, solve the equation. Check your solution(s). (See Examples 3 and 4.)

- $x - 6 = \sqrt{3x}$
- $x - 10 = \sqrt{9x}$
- $\sqrt{44 - 2x} = x - 10$
- $\sqrt{2x + 30} = x + 3$
- $\sqrt[3]{8x^3 - 1} = 2x - 1$
- $\sqrt[4]{3 - 8x^2} = 2x$
- $\sqrt{4x+1} = \sqrt{x+10}$
- $\sqrt{3x-3} - \sqrt{x+12} = 0$
- $\sqrt[3]{2x-5} - \sqrt[3]{8x+1} = 0$
- $\sqrt[3]{x+5} = 2\sqrt[3]{2x+6}$
- $\sqrt{3x-8} + 1 = \sqrt{x+5}$
- $\sqrt{x+2} = 2 - \sqrt{x}$

In Exercises 27–34, solve the equation. Check your solution(s). (See Examples 5 and 6.)

- $2x^{2/3} = 8$
- $4x^{3/2} = 32$
- $x^{1/4} + 3 = 0$
- $2x^{3/4} - 14 = 40$
- $(x+6)^{1/2} = x$
- $(5-x)^{1/2} - 2x = 0$
- $2(x+11)^{1/2} = x+3$
- $(5x^2-4)^{1/4} = x$

ERROR ANALYSIS In Exercises 35 and 36, describe and correct the error in solving the equation.

35. $\sqrt[3]{3x-8} = 4$
 $(\sqrt[3]{3x-8})^3 = 4^3$
 $3x-8 = 4$
 $3x = 12$
 $x = 4$

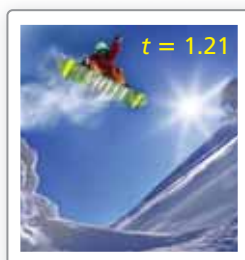
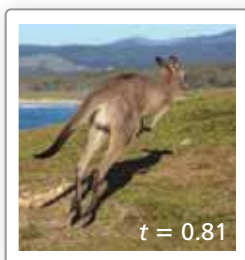
36. $8x^{3/2} = 1000$
 $8(x^{3/2})^{2/3} = 1000^{2/3}$
 $8x = 100$
 $x = \frac{25}{2}$

In Exercises 37–44, solve the inequality. (See Example 7.)

37. $2\sqrt[3]{x} - 5 \geq 3$ 38. $\sqrt[3]{x - 4} \leq 5$
 39. $4\sqrt{x - 2} > 20$ 40. $7\sqrt{x} + 1 < 9$
 41. $2\sqrt{x} + 3 \leq 8$
 42. $\sqrt[3]{x + 7} \geq 3$
 43. $-2\sqrt[3]{x + 4} < 12$
 44. $-0.25\sqrt{x} - 6 \leq -3$

45. **MODELING WITH MATHEMATICS** The length ℓ (in inches) of a standard nail can be modeled by $\ell = 54d^{3/2}$, where d is the diameter (in inches) of the nail. What is the diameter of a standard nail that is 3 inches long?

46. **DRAWING CONCLUSIONS** “Hang time” is the time you are suspended in the air during a jump. Your hang time t (in seconds) is given by the function $t = 0.5\sqrt{h}$, where h is the height (in feet) of the jump. Suppose a kangaroo and a snowboarder jump with the hang times shown.



- Find the heights that the snowboarder and the kangaroo jump.
- Double the hang times of the snowboarder and the kangaroo and calculate the corresponding heights of each jump.
- When the hang time doubles, does the height of the jump double? Explain.

USING TOOLS In Exercises 47–52, solve the nonlinear system. Justify your answer with a graph.

47. $y^2 = x - 3$ 48. $y^2 = 4x + 17$
 $y = x - 3$ $y = x + 5$
 49. $x^2 + y^2 = 4$ 50. $x^2 + y^2 = 25$
 $y = x - 2$ $y = -\frac{3}{4}x + \frac{25}{4}$
 51. $x^2 + y^2 = 1$ 52. $x^2 + y^2 = 4$
 $y = \frac{1}{2}x^2 - 1$ $y^2 = x + 2$

53. **PROBLEM SOLVING** The speed s (in miles per hour) of a car can be given by $s = \sqrt{30fd}$, where f is the coefficient of friction and d is the stopping distance (in feet). The table shows the coefficient of friction for different surfaces.

Surface	Coefficient of friction, f
dry asphalt	0.75
wet asphalt	0.30
snow	0.30
ice	0.15

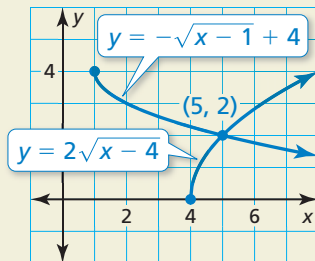
- Compare the stopping distances of a car traveling 45 miles per hour on the surfaces given in the table.
 - You are driving 35 miles per hour on an icy road when a deer jumps in front of your car. How far away must you begin to brake to avoid hitting the deer? Justify your answer.
54. **MODELING WITH MATHEMATICS** The Beaufort wind scale was devised to measure wind speed. The Beaufort numbers B , which range from 0 to 12, can be modeled by $B = 1.69\sqrt{s + 4.25} - 3.55$, where s is the wind speed (in miles per hour).

Beaufort number	Force of wind
0	calm
3	gentle breeze
6	strong breeze
9	strong gale
12	hurricane

- What is the wind speed for $B = 0$? $B = 3$?
 - Write an inequality that describes the range of wind speeds represented by the Beaufort model.
55. **USING TOOLS** Solve the equation $x - 4 = \sqrt{2x}$. Then solve the equation $x - 4 = -\sqrt{2x}$.
- How does changing $\sqrt{2x}$ to $-\sqrt{2x}$ change the solution(s) of the equation?
 - Justify your answer in part (a) using graphs.
56. **MAKING AN ARGUMENT** Your friend says it is impossible for a radical equation to have two extraneous solutions. Is your friend correct? Explain your reasoning.

57. **USING STRUCTURE** Explain how you know the radical equation $\sqrt{x+4} = -5$ has no real solution without solving it.

58. **HOW DO YOU SEE IT?** Use the graph to find the solution of the equation $2\sqrt{x-4} = -\sqrt{x-1} + 4$. Explain your reasoning.



59. **WRITING** A company determines that the price p of a product can be modeled by $p = 70 - \sqrt{0.02x + 1}$, where x is the number of units of the product demanded per day. Describe the effect that raising the price has on the number of units demanded.

60. **THOUGHT PROVOKING** City officials rope off a circular area to prepare for a concert in the park. They estimate that each person occupies 6 square feet. Describe how you can use a radical inequality to determine the possible radius of the region when P people are expected to attend the concert.



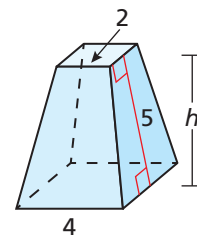
61. **MATHEMATICAL CONNECTIONS** The Moeraki Boulders along the coast of New Zealand are stone spheres with radii of approximately 3 feet. A formula for the radius of a sphere is

$$r = \frac{1}{2} \sqrt{\frac{S}{\pi}}$$

where S is the surface area of the sphere. Find the surface area of a Moeraki Boulder.

62. **PROBLEM SOLVING** You are trying to determine the height of a truncated pyramid, which cannot be measured directly. The height h and slant height ℓ of the truncated pyramid are related by the formula below.

$$\ell = \sqrt{h^2 + \frac{1}{4}(b_2 - b_1)^2}$$



In the given formula, b_1 and b_2 are the side lengths of the upper and lower bases of the pyramid, respectively. When $\ell = 5$, $b_1 = 2$, and $b_2 = 4$, what is the height of the pyramid?

63. **REWRITING A FORMULA** A burning candle has a radius of r inches and was initially h_0 inches tall. After t minutes, the height of the candle has been reduced to h inches. These quantities are related by the formula

$$r = \sqrt{\frac{kt}{\pi(h_0 - h)}}$$

where k is a constant. Suppose the radius of a candle is 0.875 inch, its initial height is 6.5 inches, and $k = 0.04$.

- Rewrite the formula, solving for h in terms of t .
- Use your formula in part (a) to determine the height of the candle after burning 45 minutes.

Maintaining Mathematical Proficiency Reviewing what you learned in previous grades and lessons

Perform the indicated operation. (Section 4.2 and Section 4.3)

64. $(x^3 - 2x^2 + 3x + 1) + (x^4 - 7x)$

65. $(2x^5 + x^4 - 4x^2) - (x^5 - 3)$

66. $(x^3 + 2x^2 + 1)(x^2 + 5)$

67. $(x^4 + 2x^3 + 11x^2 + 14x - 16) \div (x + 2)$

Let $f(x) = x^3 - 4x^2 + 6$. Write a rule for g . Describe the graph of g as a transformation of the graph of f . (Section 4.7)

68. $g(x) = f(-x) + 4$

69. $g(x) = \frac{1}{2}f(x) - 3$

70. $g(x) = -f(x - 1) + 6$

5.5 Performing Function Operations

Essential Question How can you use the graphs of two functions to sketch the graph of an arithmetic combination of the two functions?

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two functions can be combined to form other functions. For example, the functions $f(x) = 2x - 3$ and $g(x) = x^2 - 1$ can be combined to form the sum, difference, product, or quotient of f and g .

$$f(x) + g(x) = (2x - 3) + (x^2 - 1) = x^2 + 2x - 4 \quad \text{sum}$$

$$f(x) - g(x) = (2x - 3) - (x^2 - 1) = -x^2 + 2x - 2 \quad \text{difference}$$

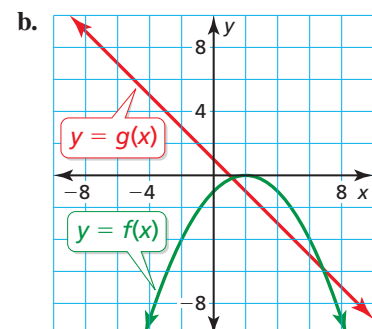
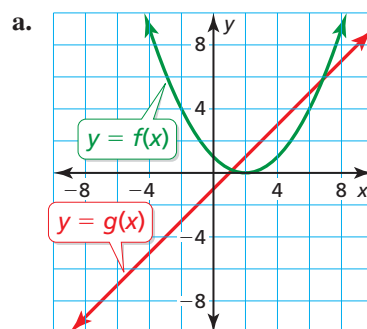
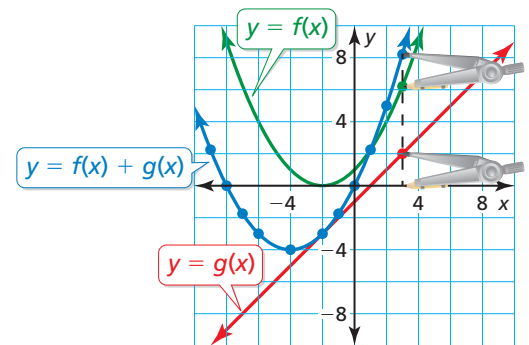
$$f(x) \cdot g(x) = (2x - 3)(x^2 - 1) = 2x^3 - 3x^2 - 2x + 3 \quad \text{product}$$

$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 - 1} \quad \text{quotient}$$

EXPLORATION 1 Graphing the Sum of Two Functions

Work with a partner. Use the graphs of f and g to sketch the graph of $f + g$. Explain your steps.

Sample Choose a point on the graph of g . Use a compass or a ruler to measure its distance above or below the x -axis. If above, add the distance to the y -coordinate of the point with the same x -coordinate on the graph of f . If below, subtract the distance. Plot the new point. Repeat this process for several points. Finally, draw a smooth curve through the new points to obtain the graph of $f + g$.



MAKING SENSE OF PROBLEMS

To be proficient in math, you need to check your answers to problems using a different method and continually ask yourself, "Does this make sense?"

Communicate Your Answer

- How can you use the graphs of two functions to sketch the graph of an arithmetic combination of the two functions?
- Check your answers in Exploration 1 by writing equations for f and g , adding the functions, and graphing the sum.

5.5 Lesson

What You Will Learn

- ▶ Add, subtract, multiply, and divide functions.

Core Vocabulary

Previous

domain

scientific notation

Operations on Functions

You have learned how to add, subtract, multiply, and divide polynomial expressions. These operations can also be defined for functions.

Core Concept

Operations on Functions

Let f and g be any two functions. A new function can be defined by performing any of the four basic operations on f and g .

Operation	Definition	Example: $f(x) = 5x, g(x) = x + 2$
Addition	$(f + g)(x) = f(x) + g(x)$	$(f + g)(x) = 5x + (x + 2) = 6x + 2$
Subtraction	$(f - g)(x) = f(x) - g(x)$	$(f - g)(x) = 5x - (x + 2) = 4x - 2$
Multiplication	$(fg)(x) = f(x) \cdot g(x)$	$(fg)(x) = 5x(x + 2) = 5x^2 + 10x$
Division	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	$\left(\frac{f}{g}\right)(x) = \frac{5x}{x + 2}$

The domains of the sum, difference, product, and quotient functions consist of the x -values that are in the domains of both f and g . Additionally, the domain of the quotient does not include x -values for which $g(x) = 0$.

EXAMPLE 1 Adding Two Functions

Let $f(x) = 3\sqrt{x}$ and $g(x) = -10\sqrt{x}$. Find $(f + g)(x)$ and state the domain. Then evaluate the sum when $x = 4$.

SOLUTION

$$(f + g)(x) = f(x) + g(x) = 3\sqrt{x} + (-10\sqrt{x}) = (3 - 10)\sqrt{x} = -7\sqrt{x}$$

The functions f and g each have the same domain: all nonnegative real numbers. So, the domain of $f + g$ also consists of all nonnegative real numbers. To evaluate $f + g$ when $x = 4$, you can use several methods. Here are two:

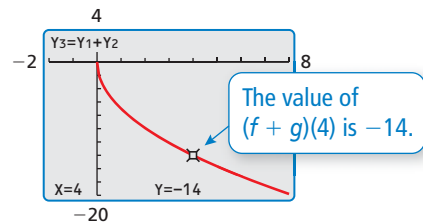
Method 1 Use an algebraic approach.

When $x = 4$, the value of the sum is

$$(f + g)(4) = -7\sqrt{4} = -14.$$

Method 2 Use a graphical approach.

Enter the functions $y_1 = 3\sqrt{x}$, $y_2 = -10\sqrt{x}$, and $y_3 = y_1 + y_2$ in a graphing calculator. Then graph y_3 , the sum of the two functions. Use the *trace* feature to find the value of $f + g$ when $x = 4$. From the graph, $(f + g)(4) = -14$.



EXAMPLE 2 Subtracting Two Functions

Let $f(x) = 3x^3 - 2x^2 + 5$ and $g(x) = x^3 - 3x^2 + 4x - 2$. Find $(f - g)(x)$ and state the domain. Then evaluate the difference when $x = -2$.

SOLUTION

$$(f - g)(x) = f(x) - g(x) = 3x^3 - 2x^2 + 5 - (x^3 - 3x^2 + 4x - 2) = 2x^3 + x^2 - 4x + 7$$

The functions f and g each have the same domain: all real numbers. So, the domain of $f - g$ also consists of all real numbers. When $x = -2$, the value of the difference is

$$(f - g)(-2) = 2(-2)^3 + (-2)^2 - 4(-2) + 7 = 3.$$

EXAMPLE 3 Multiplying Two Functions

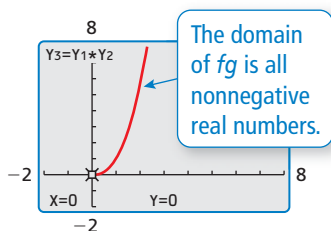
Let $f(x) = x^2$ and $g(x) = \sqrt{x}$. Find $(fg)(x)$ and state the domain. Then evaluate the product when $x = 9$.

SOLUTION

$$(fg)(x) = f(x) \cdot g(x) = x^2(\sqrt{x}) = x^2(x^{1/2}) = x^{2+1/2} = x^{5/2}$$

The domain of f consists of all real numbers, and the domain of g consists of all nonnegative real numbers. So, the domain of fg consists of all nonnegative real numbers. To confirm this, enter the functions $y_1 = x^2$, $y_2 = \sqrt{x}$, and $y_3 = y_1 \cdot y_2$ in a graphing calculator. Then graph y_3 , the product of the two functions. It appears from the graph that the domain of fg consists of all nonnegative real numbers. When $x = 9$, the value of the product is

$$(fg)(9) = 9^{5/2} = (9^{1/2})^5 = 3^5 = 243.$$



EXAMPLE 4 Dividing Two Functions

Let $f(x) = 6x$ and $g(x) = x^{3/4}$. Find $\left(\frac{f}{g}\right)(x)$ and state the domain. Then evaluate the quotient when $x = 16$.

SOLUTION

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{6x}{x^{3/4}} = 6x^{(1-3/4)} = 6x^{1/4}$$

The domain of f consists of all real numbers, and the domain of g consists of all nonnegative real numbers. Because $g(0) = 0$, the domain of $\frac{f}{g}$ is restricted to all positive real numbers. When $x = 16$, the value of the quotient is

$$\left(\frac{f}{g}\right)(16) = 6(16)^{1/4} = 6(2^4)^{1/4} = 12.$$

ANOTHER WAY

In Example 4, you can also

evaluate $\left(\frac{f}{g}\right)(16)$ as

$$\begin{aligned} \left(\frac{f}{g}\right)(16) &= \frac{f(16)}{g(16)} \\ &= \frac{6(16)}{(16)^{3/4}} \\ &= \frac{96}{8} \\ &= 12. \end{aligned}$$

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- Let $f(x) = -2x^{2/3}$ and $g(x) = 7x^{2/3}$. Find $(f + g)(x)$ and $(f - g)(x)$ and state the domain of each. Then evaluate $(f + g)(8)$ and $(f - g)(8)$.
- Let $f(x) = 3x$ and $g(x) = x^{1/5}$. Find $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$ and state the domain of each. Then evaluate $(fg)(32)$ and $\left(\frac{f}{g}\right)(32)$.

EXAMPLE 5 Performing Function Operations Using Technology

Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{9 - x^2}$. Use a graphing calculator to evaluate $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, and $\left(\frac{f}{g}\right)(x)$ when $x = 2$. Round your answers to two decimal places.

SOLUTION

Enter the functions $y_1 = \sqrt{x}$ and $y_2 = \sqrt{9 - x^2}$ in a graphing calculator. On the home screen, enter $y_1(2) + y_2(2)$. The first entry on the screen shows that $y_1(2) + y_2(2) \approx 3.65$, so $(f + g)(2) \approx 3.65$. Enter the other function operations as shown. Here are the results of the other function operations rounded to two decimal places:

```
Y1(2)+Y2(2) 3.65028154
Y1(2)-Y2(2) -.8218544151
Y1(2)*Y2(2) 3.16227766
Y1(2)/Y2(2) .632455532
```

$$(f - g)(2) \approx -0.82 \quad (fg)(2) \approx 3.16 \quad \left(\frac{f}{g}\right)(2) \approx 0.63$$

EXAMPLE 6 Solving a Real-Life Problem

For a white rhino, heart rate r (in beats per minute) and life span s (in minutes) are related to body mass m (in kilograms) by the functions

$$r(m) = 241m^{-0.25}$$

and

$$s(m) = (6 \times 10^6)m^{0.2}.$$

- Find $(rs)(m)$.
- Explain what $(rs)(m)$ represents.



SOLUTION

$$\begin{aligned} \text{a. } (rs)(m) &= r(m) \cdot s(m) \\ &= 241m^{-0.25}[(6 \times 10^6)m^{0.2}] \\ &= 241(6 \times 10^6)m^{-0.25+0.2} \\ &= (1446 \times 10^6)m^{-0.05} \\ &= (1.446 \times 10^9)m^{-0.05} \end{aligned}$$

Definition of multiplication
Write product of $r(m)$ and $s(m)$.
Product of Powers Property
Simplify.
Use scientific notation.

- Multiplying heart rate by life span gives the total number of heartbeats over the lifetime of a white rhino with body mass m .

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- Let $f(x) = 8x$ and $g(x) = 2x^{5/6}$. Use a graphing calculator to evaluate $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, and $\left(\frac{f}{g}\right)(x)$ when $x = 5$. Round your answers to two decimal places.
- In Example 5, explain why you can evaluate $(f + g)(3)$, $(f - g)(3)$, and $(fg)(3)$ but not $\left(\frac{f}{g}\right)(3)$.
- Use the answer in Example 6(a) to find the total number of heartbeats over the lifetime of a white rhino when its body mass is 1.7×10^5 kilograms.

Vocabulary and Core Concept Check

- WRITING** Let f and g be any two functions. Describe how you can use f , g , and the four basic operations to create new functions.
- WRITING** What x -values are not included in the domain of the quotient of two functions?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, find $(f + g)(x)$ and $(f - g)(x)$ and state the domain of each. Then evaluate $f + g$ and $f - g$ for the given value of x . (See Examples 1 and 2.)

- $f(x) = -5\sqrt[4]{x}$, $g(x) = 19\sqrt[4]{x}$; $x = 16$
- $f(x) = \sqrt[3]{2x}$, $g(x) = -11\sqrt[3]{2x}$; $x = -4$
- $f(x) = 6x - 4x^2 - 7x^3$, $g(x) = 9x^2 - 5x$; $x = -1$
- $f(x) = 11x + 2x^2$, $g(x) = -7x - 3x^2 + 4$; $x = 2$

In Exercises 7–12, find $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$ and state the domain of each. Then evaluate fg and $\frac{f}{g}$ for the given value of x . (See Examples 3 and 4.)

- $f(x) = 2x^3$, $g(x) = \sqrt[3]{x}$; $x = -27$
- $f(x) = x^4$, $g(x) = 3\sqrt{x}$; $x = 4$
- $f(x) = 4x$, $g(x) = 9x^{1/2}$; $x = 9$
- $f(x) = 11x^3$, $g(x) = 7x^{7/3}$; $x = -8$
- $f(x) = 7x^{3/2}$, $g(x) = -14x^{1/3}$; $x = 64$
- $f(x) = 4x^{5/4}$, $g(x) = 2x^{1/2}$; $x = 16$

USING TOOLS In Exercises 13–16, use a graphing calculator to evaluate $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, and $\left(\frac{f}{g}\right)(x)$ when $x = 5$. Round your answers to two decimal places. (See Example 5.)

- $f(x) = 4x^4$; $g(x) = 24x^{1/3}$
- $f(x) = 7x^{5/3}$; $g(x) = 49x^{2/3}$
- $f(x) = -2x^{1/3}$; $g(x) = 5x^{1/2}$
- $f(x) = 4x^{1/2}$; $g(x) = 6x^{3/4}$

ERROR ANALYSIS In Exercises 17 and 18, describe and correct the error in stating the domain.

17.



$f(x) = x^{1/2}$ and $g(x) = x^{3/2}$
The domain of fg is all real numbers.

18.



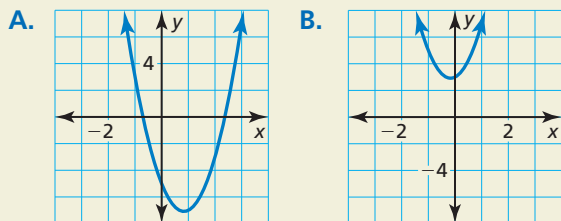
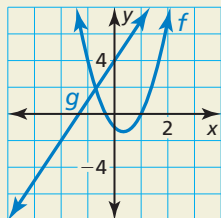
$f(x) = x^3$ and $g(x) = x^2 - 4$
The domain of $\frac{f}{g}$ is all real numbers except $x = 2$.

- MODELING WITH MATHEMATICS** From 1990 to 2010, the numbers (in millions) of female F and male M employees from the ages of 16 to 19 in the United States can be modeled by $F(t) = -0.007t^2 + 0.10t + 3.7$ and $M(t) = 0.0001t^3 - 0.009t^2 + 0.11t + 3.7$, where t is the number of years since 1990. (See Example 6.)
 - Find $(F + M)(t)$.
 - Explain what $(F + M)(t)$ represents.
- MODELING WITH MATHEMATICS** From 2005 to 2009, the numbers of cruise ship departures (in thousands) from around the world W and Florida F can be modeled by the equations

$$W(t) = -5.8333t^3 + 17.43t^2 + 509.1t + 11496$$

$$F(t) = 12.5t^3 - 60.29t^2 + 136.6t + 4881$$
 where t is the number of years since 2005.
 - Find $(W - F)(t)$.
 - Explain what $(W - F)(t)$ represents.
- MAKING AN ARGUMENT** Your friend claims that the addition of functions and the multiplication of functions are commutative. Is your friend correct? Explain your reasoning.

22. **HOW DO YOU SEE IT?** The graphs of the functions $f(x) = 3x^2 - 2x - 1$ and $g(x) = 3x + 4$ are shown. Which graph represents the function $f + g$? the function $f - g$? Explain your reasoning.



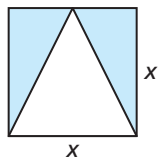
23. **REASONING** The table shows the outputs of the two functions f and g . Use the table to evaluate $(f + g)(3)$, $(f - g)(1)$, $(fg)(2)$, and $(\frac{f}{g})(0)$.

x	0	1	2	3	4
$f(x)$	-2	-4	0	10	26
$g(x)$	-1	-3	-13	-31	-57

24. **THOUGHT PROVOKING** Is it possible to write two functions whose sum contains radicals, but whose product does not? Justify your answers.

25. **MATHEMATICAL CONNECTIONS**

A triangle is inscribed in a square, as shown. Write and simplify a function r in terms of x that represents the area of the shaded region.



26. **REWRITING A FORMULA** For a mammal that weighs w grams, the volume b (in milliliters) of air breathed in and the volume d (in milliliters) of “dead space” (the portion of the lungs not filled with air) can be modeled by

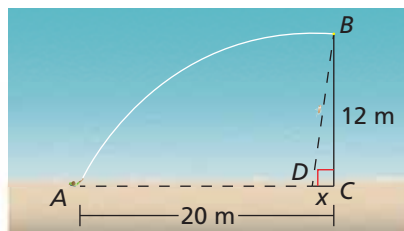
$$b(w) = 0.007w \text{ and } d(w) = 0.002w.$$

The breathing rate r (in breaths per minute) of a mammal that weighs w grams can be modeled by

$$r(w) = \frac{1.1w^{0.734}}{b(w) - d(w)}.$$

Simplify $r(w)$ and calculate the breathing rate for body weights of 6.5 grams, 300 grams, and 70,000 grams.

27. **PROBLEM SOLVING** A mathematician at a lake throws a tennis ball from point A along the water's edge to point B in the water, as shown. His dog, Elvis, first runs along the beach from point A to point D and then swims to fetch the ball at point B .



- Elvis runs at a speed of about 6.4 meters per second. Write a function r in terms of x that represents the time he spends running from point A to point D . Elvis swims at a speed of about 0.9 meter per second. Write a function s in terms of x that represents the time he spends swimming from point D to point B .
- Write a function t in terms of x that represents the total time Elvis spends traveling from point A to point D to point B .
- Use a graphing calculator to graph t . Find the value of x that minimizes t . Explain the meaning of this value.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the literal equation for n . (*Skills Review Handbook*)

28. $3xn - 9 = 6y$

29. $5z = 7n + 8nz$

30. $3nb = 5n - 6z$

31. $\frac{3 + 4n}{n} = 7b$

Determine whether the relation is a function. Explain. (*Skills Review Handbook*)

32. $(3, 4), (4, 6), (1, 4), (2, -1)$

33. $(-1, 2), (3, 7), (0, 2), (-1, -1)$

34. $(1, 6), (7, -3), (4, 0), (3, 0)$

35. $(3, 8), (2, 5), (9, 5), (2, -3)$

5.6 Inverse of a Function

Essential Question How can you sketch the graph of the inverse of a function?

EXPLORATION 1 Graphing Functions and Their Inverses

Work with a partner. Each pair of functions are *inverses* of each other. Use a graphing calculator to graph f and g in the same viewing window. What do you notice about the graphs?

a. $f(x) = 4x + 3$
 $g(x) = \frac{x - 3}{4}$

b. $f(x) = x^3 + 1$
 $g(x) = \sqrt[3]{x - 1}$

c. $f(x) = \sqrt{x - 3}$
 $g(x) = x^2 + 3, x \geq 0$

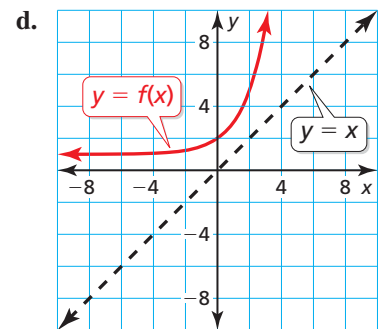
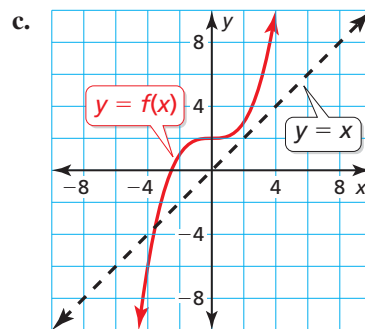
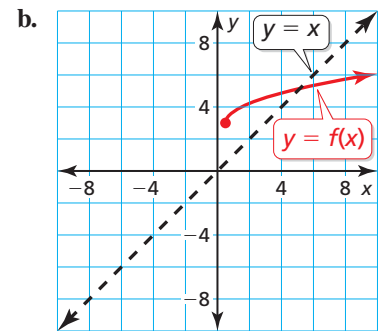
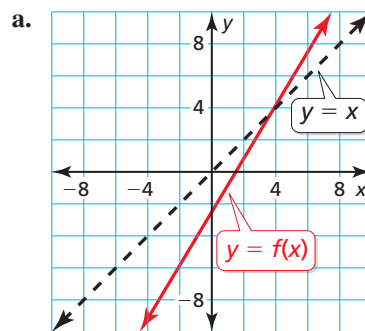
d. $f(x) = \frac{4x + 4}{x + 5}$
 $g(x) = \frac{4 - 5x}{x - 4}$

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to reason inductively and make a plausible argument.

EXPLORATION 2 Sketching Graphs of Inverse Functions

Work with a partner. Use the graph of f to sketch the graph of g , the inverse function of f , on the same set of coordinate axes. Explain your reasoning.



Communicate Your Answer

- How can you sketch the graph of the inverse of a function?
- In Exploration 1, what do you notice about the relationship between the equations of f and g ? Use your answer to find g , the inverse function of $f(x) = 2x - 3$.

Use a graph to check your answer.

5.6 Lesson

Core Vocabulary

inverse functions, p. 277

Previous

- input
- output
- inverse operations
- reflection
- line of reflection

What You Will Learn

- ▶ Explore inverses of functions.
- ▶ Find and verify inverses of nonlinear functions.
- ▶ Solve real-life problems using inverse functions.

Exploring Inverses of Functions

You have used given inputs to find corresponding outputs of $y = f(x)$ for various types of functions. You have also used given outputs to find corresponding inputs. Now you will solve equations of the form $y = f(x)$ for x to obtain a general formula for finding the input given a specific output of a function f .

EXAMPLE 1 Writing a Formula for the Input of a Function

Let $f(x) = 2x + 3$.

- Solve $y = f(x)$ for x .
- Find the input when the output is -7 .

SOLUTION

$$\begin{aligned} \text{a.} \quad y &= 2x + 3 && \text{Set } y \text{ equal to } f(x). \\ y - 3 &= 2x && \text{Subtract 3 from each side.} \\ \frac{y - 3}{2} &= x && \text{Divide each side by 2.} \end{aligned}$$

- Find the input when $y = -7$.

$$\begin{aligned} x &= \frac{-7 - 3}{2} && \text{Substitute } -7 \text{ for } y. \\ &= \frac{-10}{2} && \text{Subtract.} \\ &= -5 && \text{Divide.} \end{aligned}$$

▶ So, the input is -5 when the output is -7 .

Check

$$\begin{aligned} f(-5) &= 2(-5) + 3 \\ &= -10 + 3 \\ &= -7 \quad \checkmark \end{aligned}$$

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Solve $y = f(x)$ for x . Then find the input(s) when the output is 2.

- $f(x) = x - 2$
- $f(x) = 2x^2$
- $f(x) = -x^3 + 3$

In Example 1, notice the steps involved after substituting for x in $y = 2x + 3$ and after substituting for y in $x = \frac{y - 3}{2}$.

$y = 2x + 3$	$x = \frac{y - 3}{2}$
Step 1 Multiply by 2.	Step 1 Subtract 3.
Step 2 Add 3.	Step 2 Divide by 2.

inverse operations in the reverse order

UNDERSTANDING MATHEMATICAL TERMS

The term *inverse functions* does not refer to a new type of function. Rather, it describes any pair of functions that are inverses.

Notice that these steps *undo* each other. Functions that undo each other are called **inverse functions**. In Example 1, you can use the equation solved for x to write the inverse of f by switching the roles of x and y .

$$f(x) = 2x + 3 \quad \text{original function} \qquad g(x) = \frac{x - 3}{2} \quad \text{inverse function}$$

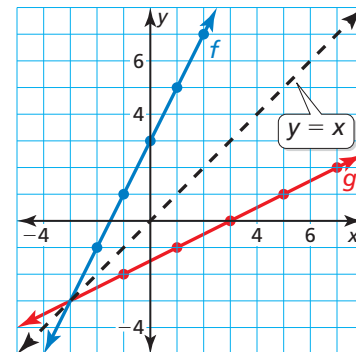
Because inverse functions interchange the input and output values of the original function, the domain and range are also interchanged.

Original function: $f(x) = 2x + 3$

x	-2	-1	0	1	2
y	-1	1	3	5	7

Inverse function: $g(x) = \frac{x - 3}{2}$

x	-1	1	3	5	7
y	-2	-1	0	1	2



The graph of an inverse function is a *reflection* of the graph of the original function. The *line of reflection* is $y = x$. To find the inverse of a function algebraically, switch the roles of x and y , and then solve for y .

EXAMPLE 2 Finding the Inverse of a Linear Function

Find the inverse of $f(x) = 3x - 1$.

SOLUTION

Method 1 Use inverse operations in the reverse order.

$$f(x) = 3x - 1 \quad \text{Multiply the input } x \text{ by 3 and then subtract 1.}$$

To find the inverse, apply inverse operations in the reverse order.

$$g(x) = \frac{x + 1}{3} \quad \text{Add 1 to the input } x \text{ and then divide by 3.}$$

▶ The inverse of f is $g(x) = \frac{x + 1}{3}$, or $g(x) = \frac{1}{3}x + \frac{1}{3}$.

Method 2 Set y equal to $f(x)$. Switch the roles of x and y and solve for y .

$$y = 3x - 1 \quad \text{Set } y \text{ equal to } f(x).$$

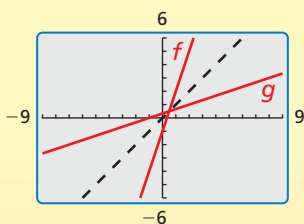
$$x = 3y - 1 \quad \text{Switch } x \text{ and } y.$$

$$x + 1 = 3y \quad \text{Add 1 to each side.}$$

$$\frac{x + 1}{3} = y \quad \text{Divide each side by 3.}$$

▶ The inverse of f is $g(x) = \frac{x + 1}{3}$, or $g(x) = \frac{1}{3}x + \frac{1}{3}$.

Check



The graph of g appears to be a reflection of the graph of f in the line $y = x$. ✓

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Find the inverse of the function. Then graph the function and its inverse.

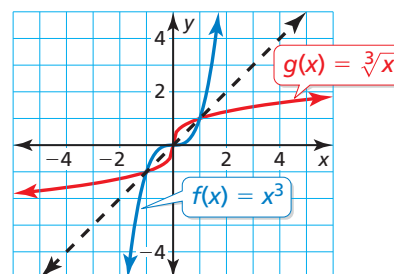
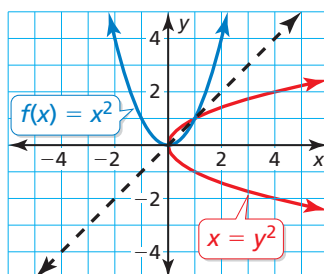
4. $f(x) = 2x$

5. $f(x) = -x + 1$

6. $f(x) = \frac{1}{3}x - 2$

Inverses of Nonlinear Functions

In the previous examples, the inverses of the linear functions were also functions. However, inverses are not always functions. The graphs of $f(x) = x^2$ and $f(x) = x^3$ are shown along with their reflections in the line $y = x$. Notice that the inverse of $f(x) = x^3$ is a function, but the inverse of $f(x) = x^2$ is *not* a function.



When the domain of $f(x) = x^2$ is *restricted* to only nonnegative real numbers, the inverse of f is a function.

EXAMPLE 3 Finding the Inverse of a Quadratic Function

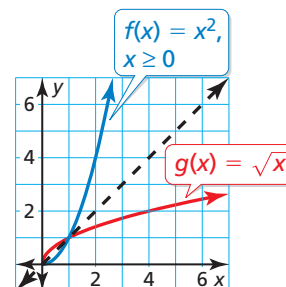
Find the inverse of $f(x) = x^2, x \geq 0$. Then graph the function and its inverse.

SOLUTION

$f(x) = x^2$	Write the original function.
$y = x^2$	Set y equal to $f(x)$.
$x = y^2$	Switch x and y .
$\pm\sqrt{x} = y$	Take square root of each side.

The domain of f is restricted to nonnegative values of x . So, the range of the inverse must also be restricted to nonnegative values.

▶ So, the inverse of f is $g(x) = \sqrt{x}$.



STUDY TIP

If the domain of f were restricted to $x \leq 0$, then the inverse would be $g(x) = -\sqrt{x}$.

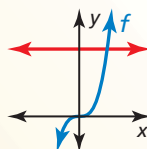
You can use the graph of a function f to determine whether the inverse of f is a function by applying the *horizontal line test*.

Core Concept

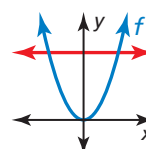
Horizontal Line Test

The inverse of a function f is also a function if and only if no horizontal line intersects the graph of f more than once.

Inverse is a function



Inverse is not a function

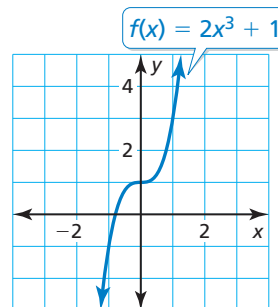


EXAMPLE 4 Finding the Inverse of a Cubic Function

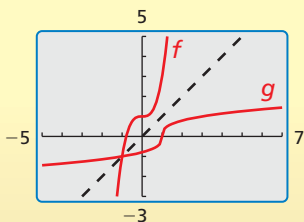
Consider the function $f(x) = 2x^3 + 1$. Determine whether the inverse of f is a function. Then find the inverse.

SOLUTION

Graph the function f . Notice that no horizontal line intersects the graph more than once. So, the inverse of f is a function. Find the inverse.



Check



$$y = 2x^3 + 1$$

Set y equal to $f(x)$.

$$x = 2y^3 + 1$$

Switch x and y .

$$x - 1 = 2y^3$$

Subtract 1 from each side.

$$\frac{x - 1}{2} = y^3$$

Divide each side by 2.

$$\sqrt[3]{\frac{x - 1}{2}} = y$$

Take cube root of each side.

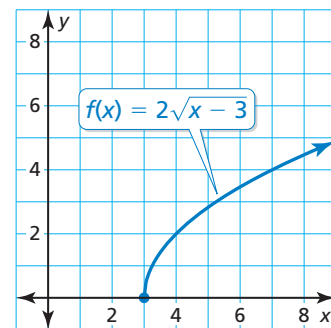
► So, the inverse of f is $g(x) = \sqrt[3]{\frac{x - 1}{2}}$.

EXAMPLE 5 Finding the Inverse of a Radical Function

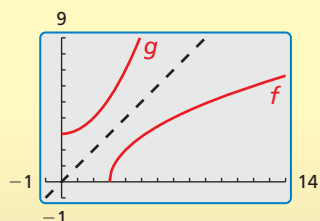
Consider the function $f(x) = 2\sqrt{x - 3}$. Determine whether the inverse of f is a function. Then find the inverse.

SOLUTION

Graph the function f . Notice that no horizontal line intersects the graph more than once. So, the inverse of f is a function. Find the inverse.



Check



$$y = 2\sqrt{x - 3}$$

Set y equal to $f(x)$.

$$x = 2\sqrt{y - 3}$$

Switch x and y .

$$x^2 = (2\sqrt{y - 3})^2$$

Square each side.

$$x^2 = 4(y - 3)$$

Simplify.

$$x^2 = 4y - 12$$

Distributive Property

$$x^2 + 12 = 4y$$

Add 12 to each side.

$$\frac{1}{4}x^2 + 3 = y$$

Divide each side by 4.

Because the range of f is $y \geq 0$, the domain of the inverse must be restricted to $x \geq 0$.

► So, the inverse of f is $g(x) = \frac{1}{4}x^2 + 3$, where $x \geq 0$.

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Find the inverse of the function. Then graph the function and its inverse.

7. $f(x) = -x^2, x \leq 0$

8. $f(x) = -x^3 + 4$

9. $f(x) = \sqrt{x + 2}$

REASONING ABSTRACTLY

Inverse functions *undo* each other. So, when you evaluate a function for a specific input, and then evaluate its inverse using the output, you obtain the original input.

Let f and g be inverse functions. If $f(a) = b$, then $g(b) = a$. So, in general,

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x.$$

EXAMPLE 6 Verifying Functions Are Inverses

Verify that $f(x) = 3x - 1$ and $g(x) = \frac{x + 1}{3}$ are inverse functions.

SOLUTION

Step 1 Show that $f(g(x)) = x$.

$$\begin{aligned} f(g(x)) &= f\left(\frac{x + 1}{3}\right) \\ &= 3\left(\frac{x + 1}{3}\right) - 1 \\ &= x + 1 - 1 \\ &= x \quad \checkmark \end{aligned}$$

Step 2 Show that $g(f(x)) = x$.

$$\begin{aligned} g(f(x)) &= g(3x - 1) \\ &= \frac{3x - 1 + 1}{3} \\ &= \frac{3x}{3} \\ &= x \quad \checkmark \end{aligned}$$

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Determine whether the functions are inverse functions.

10. $f(x) = x + 5$, $g(x) = x - 5$

11. $f(x) = 8x^3$, $g(x) = \sqrt[3]{2x}$

Solving Real-Life Problems

In many real-life problems, formulas contain meaningful variables, such as the radius r in the formula for the surface area S of a sphere, $S = 4\pi r^2$. In this situation, switching the variables to find the inverse would create confusion by switching the meanings of S and r . So, when finding the inverse, solve for r without switching the variables.

EXAMPLE 7 Solving a Multi-Step Problem

Find the inverse of the function that represents the surface area of a sphere, $S = 4\pi r^2$. Then find the radius of a sphere that has a surface area of 100π square feet.

SOLUTION

Step 1 Find the inverse of the function.

$$S = 4\pi r^2$$

$$\frac{S}{4\pi} = r^2$$

$$\sqrt{\frac{S}{4\pi}} = r$$

The radius r must be positive, so disregard the negative square root.

Step 2 Evaluate the inverse when

$$S = 100\pi.$$

$$r = \sqrt{\frac{100\pi}{4\pi}}$$

$$= \sqrt{25} = 5$$

▶ The radius of the sphere is 5 feet.

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12. The distance d (in meters) that a dropped object falls in t seconds on Earth is represented by $d = 4.9t^2$. Find the inverse of the function. How long does it take an object to fall 50 meters?

5.6 Exercises

Vocabulary and Core Concept Check

- VOCABULARY** In your own words, state the definition of inverse functions.
- WRITING** Explain how to determine whether the inverse of a function is also a function.
- COMPLETE THE SENTENCE** Functions f and g are inverses of each other provided that $f(g(x)) = \underline{\hspace{2cm}}$ and $g(f(x)) = \underline{\hspace{2cm}}$.
- DIFFERENT WORDS, SAME QUESTION** Which is different? Find “both” answers.

Let $f(x) = 5x - 2$. Solve $y = f(x)$ for x and then switch the roles of x and y .

Write an equation that represents a reflection of the graph of $f(x) = 5x - 2$ in the x -axis.

Write an equation that represents a reflection of the graph of $f(x) = 5x - 2$ in the line $y = x$.

Find the inverse of $f(x) = 5x - 2$.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, solve $y = f(x)$ for x . Then find the input(s) when the output is -3 . (See Example 1.)

- $f(x) = 3x + 5$
- $f(x) = -7x - 2$
- $f(x) = \frac{1}{2}x - 3$
- $f(x) = -\frac{2}{3}x + 1$
- $f(x) = 3x^3$
- $f(x) = 2x^4 - 5$
- $f(x) = (x - 2)^2 - 7$
- $f(x) = (x - 5)^3 - 1$

In Exercises 13–20, find the inverse of the function. Then graph the function and its inverse. (See Example 2.)

- $f(x) = 6x$
- $f(x) = -3x$
- $f(x) = -2x + 5$
- $f(x) = 6x - 3$
- $f(x) = -\frac{1}{2}x + 4$
- $f(x) = \frac{1}{3}x - 1$
- $f(x) = \frac{2}{3}x - \frac{1}{3}$
- $f(x) = -\frac{4}{5}x + \frac{1}{5}$

- COMPARING METHODS** Find the inverse of the function $f(x) = -3x + 4$ by switching the roles of x and y and solving for y . Then find the inverse of the function f by using inverse operations in the reverse order. Which method do you prefer? Explain.

- REASONING** Determine whether each pair of functions f and g are inverses. Explain your reasoning.

a.

x	-2	-1	0	1	2
$f(x)$	-2	1	4	7	10

x	-2	1	4	7	10
$g(x)$	-2	-1	0	1	2

b.

x	2	3	4	5	6
$f(x)$	8	6	4	2	0

x	2	3	4	5	6
$g(x)$	-8	-6	-4	-2	0

c.


x	-4	-2	0	2	4
$f(x)$	2	10	18	26	34


x	-4	-2	0	2	4
$g(x)$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{18}$	$\frac{1}{26}$	$\frac{1}{34}$

In Exercises 23–28, find the inverse of the function. Then graph the function and its inverse. (See Example 3.)

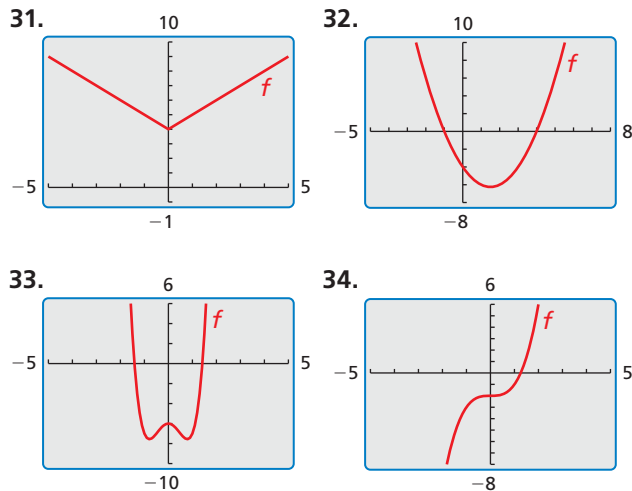
23. $f(x) = 4x^2, x \leq 0$ 24. $f(x) = 9x^2, x \leq 0$
 25. $f(x) = (x - 3)^3$ 26. $f(x) = (x + 4)^3$
 27. $f(x) = 2x^4, x \geq 0$ 28. $f(x) = -x^6, x \geq 0$

ERROR ANALYSIS In Exercises 29 and 30, describe and correct the error in finding the inverse of the function.

29.  $f(x) = -x + 3$
 $y = -x + 3$
 $-x = y + 3$
 $-x - 3 = y$

30.  $f(x) = \frac{1}{7}x^2, x \geq 0$
 $y = \frac{1}{7}x^2$
 $x = \frac{1}{7}y^2$
 $7x = y^2$
 $\pm\sqrt{7x} = y$

USING TOOLS In Exercises 31–34, use the graph to determine whether the inverse of f is a function. Explain your reasoning.



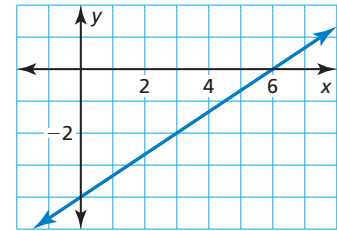
In Exercises 35–46, determine whether the inverse of f is a function. Then find the inverse. (See Examples 4 and 5.)

35. $f(x) = x^3 - 1$ 36. $f(x) = -x^3 + 3$
 37. $f(x) = \sqrt{x + 4}$ 38. $f(x) = \sqrt{x - 6}$

39. $f(x) = 2\sqrt[3]{x - 5}$ 40. $f(x) = 2x^2 - 5$
 41. $f(x) = x^4 + 2$ 42. $f(x) = 2x^3 - 5$
 43. $f(x) = 3\sqrt[3]{x + 1}$ 44. $f(x) = -\sqrt[3]{\frac{2x + 4}{3}}$
 45. $f(x) = \frac{1}{2}x^5$ 46. $f(x) = -3\sqrt{\frac{4x - 7}{3}}$

47. **WRITING EQUATIONS** What is the inverse of the function whose graph is shown?

- (A) $g(x) = \frac{3}{2}x - 6$
 (B) $g(x) = \frac{3}{2}x + 6$
 (C) $g(x) = \frac{2}{3}x - 6$
 (D) $g(x) = \frac{2}{3}x + 12$



48. **WRITING EQUATIONS** What is the inverse of $f(x) = -\frac{1}{64}x^3$?

- (A) $g(x) = -4x^3$ (B) $g(x) = 4\sqrt[3]{x}$
 (C) $g(x) = -4\sqrt[3]{x}$ (D) $g(x) = \sqrt[3]{-4x}$

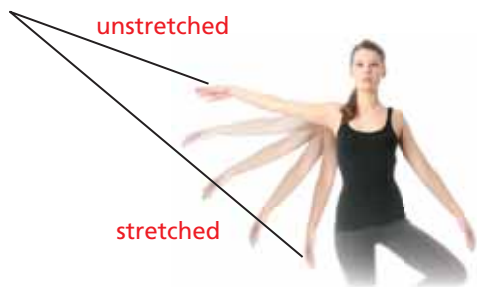
In Exercises 49–52, determine whether the functions are inverses. (See Example 6.)

49. $f(x) = 2x - 9, g(x) = \frac{x}{2} + 9$
 50. $f(x) = \frac{x - 3}{4}, g(x) = 4x + 3$
 51. $f(x) = \sqrt[5]{\frac{x + 9}{5}}, g(x) = 5x^5 - 9$
 52. $f(x) = 7x^{3/2} - 4, g(x) = \left(\frac{x + 4}{7}\right)^{3/2}$

53. **MODELING WITH MATHEMATICS** The maximum hull speed v (in knots) of a boat with a displacement hull can be approximated by $v = 1.34\sqrt{\ell}$, where ℓ is the waterline length (in feet) of the boat. Find the inverse function. What waterline length is needed to achieve a maximum speed of 7.5 knots? (See Example 7.)

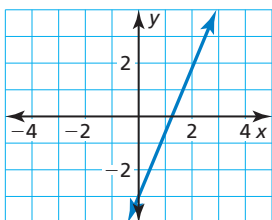


54. **MODELING WITH MATHEMATICS** Elastic bands can be used for exercising to provide a range of resistance. The resistance R (in pounds) of a band can be modeled by $R = \frac{3}{8}L - 5$, where L is the total length (in inches) of the stretched band. Find the inverse function. What length of the stretched band provides 19 pounds of resistance?

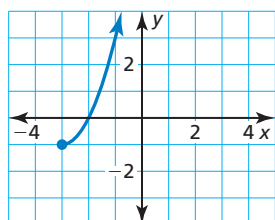


ANALYZING RELATIONSHIPS In Exercises 55–58, match the graph of the function with the graph of its inverse.

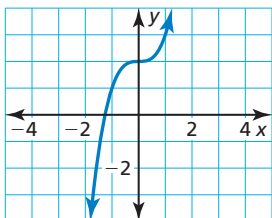
55.



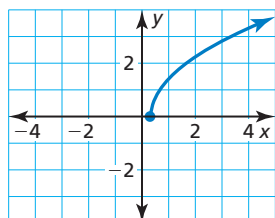
56.



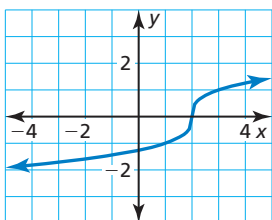
57.



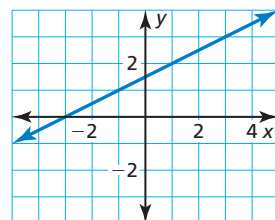
58.



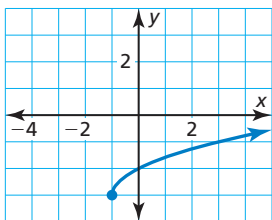
A.



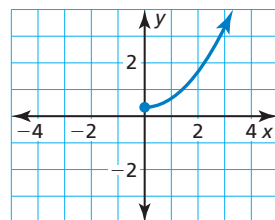
B.



C.



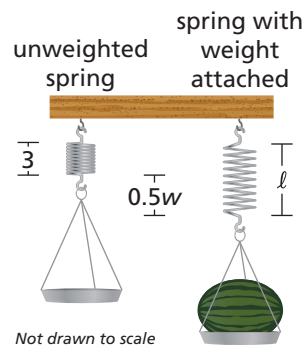
D.



59. **REASONING** You and a friend are playing a number-guessing game. You ask your friend to think of a positive number, square the number, multiply the result by 2, and then add 3. Your friend's final answer is 53. What was the original number chosen? Justify your answer.

60. **MAKING AN ARGUMENT** Your friend claims that every quadratic function whose domain is restricted to nonnegative values has an inverse function. Is your friend correct? Explain your reasoning.

61. **PROBLEM SOLVING** When calibrating a spring scale, you need to know how far the spring stretches for various weights. Hooke's Law states that the length a spring stretches is proportional to the weight attached to it. A model for one scale is $\ell = 0.5w + 3$, where ℓ is the total length (in inches) of the stretched spring and w is the weight (in pounds) of the object.



- Find the inverse function. Describe what it represents.
- You place a melon on the scale, and the spring stretches to a total length of 5.5 inches. Determine the weight of the melon.
- Verify that the function $\ell = 0.5w + 3$ and the inverse model in part (a) are inverse functions.

62. **THOUGHT PROVOKING** Do functions of the form $y = x^{m/n}$, where m and n are positive integers, have inverse functions? Justify your answer with examples.

63. **PROBLEM SOLVING** At the start of a dog sled race in Anchorage, Alaska, the temperature was 5°C . By the end of the race, the temperature was -10°C . The formula for converting temperatures from degrees Fahrenheit F to degrees Celsius C is $C = \frac{5}{9}(F - 32)$.



- Find the inverse function. Describe what it represents.
- Find the Fahrenheit temperatures at the start and end of the race.
- Use a graphing calculator to graph the original function and its inverse. Find the temperature that is the same on both temperature scales.

64. PROBLEM SOLVING The surface area A (in square meters) of a person with a mass of 60 kilograms can be approximated by $A = 0.2195h^{0.3964}$, where h is the height (in centimeters) of the person.

- Find the inverse function. Then estimate the height of a 60-kilogram person who has a body surface area of 1.6 square meters.
- Verify that function A and the inverse model in part (a) are inverse functions.

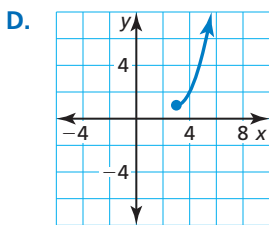
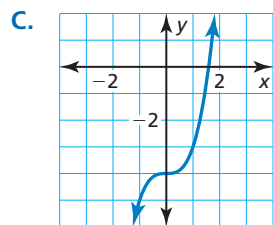
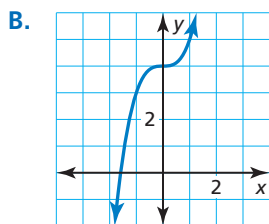
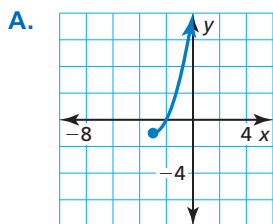
USING STRUCTURE In Exercises 65–68, match the function with the graph of its inverse.

65. $f(x) = \sqrt[3]{x - 4}$

66. $f(x) = \sqrt[3]{x + 4}$

67. $f(x) = \sqrt{x + 1} - 3$

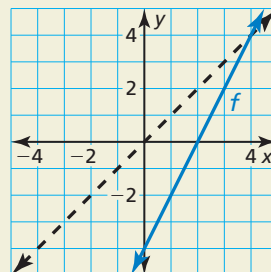
68. $f(x) = \sqrt{x - 1} + 3$



69. DRAWING CONCLUSIONS Determine whether the statement is *true* or *false*. Explain your reasoning.

- If $f(x) = x^n$ and n is a positive even integer, then the inverse of f is a function.
- If $f(x) = x^n$ and n is a positive odd integer, then the inverse of f is a function.

70. HOW DO YOU SEE IT? The graph of the function f is shown. Name three points that lie on the graph of the inverse of f . Explain your reasoning.



71. ABSTRACT REASONING Show that the inverse of any linear function $f(x) = mx + b$, where $m \neq 0$, is also a linear function. Identify the slope and y-intercept of the graph of the inverse function in terms of m and b .

72. CRITICAL THINKING Consider the function $f(x) = -x$.

- Graph $f(x) = -x$ and explain why it is its own inverse. Also, verify that $f(x) = -x$ is its own inverse algebraically.
- Graph other linear functions that are their own inverses. Write equations of the lines you graphed.
- Use your results from part (b) to write a general equation describing the family of linear functions that are their own inverses.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. Write your answer using only positive exponents. (*Skills Review Handbook*)

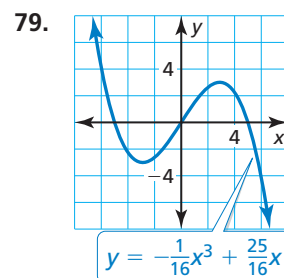
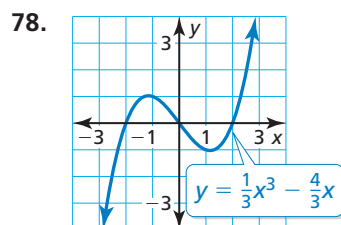
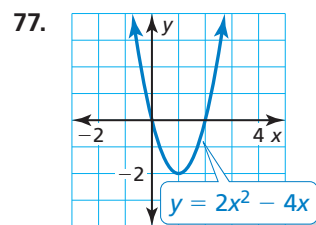
73. $(-3)^{-3}$

74. $2^3 \cdot 2^2$

75. $\frac{4^5}{4^3}$

76. $\left(\frac{2}{3}\right)^4$

Describe the x -values for which the function is increasing, decreasing, positive, and negative. (*Section 4.1*)



5.4–5.6 What Did You Learn?

Core Vocabulary

radical equation, p. 262
extraneous solutions, p. 263
inverse functions, p. 277

Core Concepts

Section 5.4

Solving Radical Equations, p. 262
Solving Radical Inequalities, p. 265

Section 5.5

Operations on Functions, p. 270

Section 5.6

Exploring Inverses of Functions, p. 276
Inverses of Nonlinear Functions, p. 278
Horizontal Line Test, p. 278

Mathematical Practices

1. How did you find the endpoints of the range in part (b) of Exercise 54 on page 267?
2. How did you use structure in Exercise 57 on page 268?
3. How can you evaluate the reasonableness of the results in Exercise 27 on page 274?
4. How can you use a graphing calculator to check your answers in Exercises 49–52 on page 282?

Performance Task

Turning the Tables

In this chapter, you have used properties of rational exponents and functions to find an answer to the problem. Using those same properties, can you find a problem to the answer? How many problems can you find?

To explore the answers to these questions and more, go to BigIdeasMath.com.



5.1 *n*th Roots and Rational Exponents (pp. 237–242)a. Evaluate $8^{4/3}$ without using a calculator.**Rational Exponent Form**

$$8^{4/3} = (8^{1/3})^4 = 2^4 = 16$$

Radical Form

$$8^{4/3} = (\sqrt[3]{8})^4 = 2^4 = 16$$

b. Find the real solution(s) of $x^4 - 45 = 580$.

$$x^4 - 45 = 580$$

$$x^4 = 625$$

$$x = \pm\sqrt[4]{625}$$

$$x = 5 \quad \text{or} \quad x = -5$$

Write original equation.

Add 45 to each side.

Take fourth root of each side.

Simplify.

► The solutions are $x = 5$ and $x = -5$.

Evaluate the expression without using a calculator.

1. $8^{7/3}$

2. $9^{5/2}$

3. $(-27)^{-2/3}$

Find the real solution(s) of the equation. Round your answer to two decimal places when appropriate.

4. $x^5 + 17 = 35$

5. $7x^3 = 189$

6. $(x + 8)^4 = 16$

5.2 Properties of Rational Exponents and Radicals (pp. 243–250)a. Use the properties of rational exponents to simplify $\left(\frac{54^{1/3}}{2^{1/3}}\right)^4$.

$$\left(\frac{54^{1/3}}{2^{1/3}}\right)^4 = \left[\left(\frac{54}{2}\right)^{1/3}\right]^4 = (27^{1/3})^4 = 3^4 = 81$$

b. Write $\sqrt[4]{16x^{13}y^8z^7}$ in simplest form.

$$\sqrt[4]{16x^{13}y^8z^7} = \sqrt[4]{16x^{12}xy^8z^4z^3}$$

$$= \sqrt[4]{16x^{12}y^8z^4} \cdot \sqrt[4]{xz^3}$$

$$= 2y^2|x^3z| \cdot \sqrt[4]{xz^3}$$

Factor out perfect fourth powers.

Product Property of Radicals

Simplify.

Simplify the expression.

7. $\left(\frac{6^{1/5}}{6^{2/5}}\right)^3$

8. $\sqrt[4]{32} \cdot \sqrt[4]{8}$

9. $\frac{1}{2 - \sqrt[4]{9}}$

10. $4\sqrt[5]{8} + 3\sqrt[5]{8}$

11. $2\sqrt{48} - \sqrt{3}$

12. $(5^{2/3} \cdot 2^{3/2})^{1/2}$

Simplify the expression. Assume all variables are positive.

13. $\sqrt[3]{125z^9}$

14. $\frac{2^{1/4}z^{5/4}}{6z}$

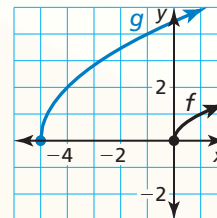
15. $\sqrt{10z^5} - z^2\sqrt{40z}$

5.3 Graphing Radical Functions (pp. 251–258)

Describe the transformation of $f(x) = \sqrt{x}$ represented by $g(x) = 2\sqrt{x+5}$. Then graph each function.

Notice that the function is of the form $g(x) = a\sqrt{x-h}$, where $a = 2$ and $h = -5$.

► So, the graph of g is a vertical stretch by a factor of 2 and a translation 5 units left of the graph of f .



Describe the transformation of f represented by g . Then graph each function.

16. $f(x) = \sqrt{x}$, $g(x) = -2\sqrt{x}$

17. $f(x) = \sqrt[3]{x}$, $g(x) = \sqrt[3]{-x} - 6$

18. Let the graph of g be a reflection in the y -axis, followed by a translation 7 units to the right of the graph of $f(x) = \sqrt[3]{x}$. Write a rule for g .

19. Use a graphing calculator to graph $2y^2 = x - 8$. Identify the vertex and the direction that the parabola opens.

20. Use a graphing calculator to graph $x^2 + y^2 = 81$. Identify the radius and the intercepts.

5.4 Solving Radical Equations and Inequalities (pp. 261–268)

Solve $6\sqrt{x+2} < 18$.

Step 1 Solve for x .

$$6\sqrt{x+2} < 18$$

Write the original inequality.

$$\sqrt{x+2} < 3$$

Divide each side by 6.

$$x+2 < 9$$

Square each side.

$$x < 7$$

Subtract 2 from each side.

Step 2 Consider the radicand.

$$x+2 \geq 0$$

The radicand cannot be negative.

$$x \geq -2$$

Subtract 2 from each side.

► So, the solution is $-2 \leq x < 7$.

Solve the equation. Check your solution.

21. $4\sqrt[3]{2x+1} = 20$

22. $\sqrt{4x-4} = \sqrt{5x-1} - 1$

23. $(6x)^{2/3} = 36$

Solve the inequality.

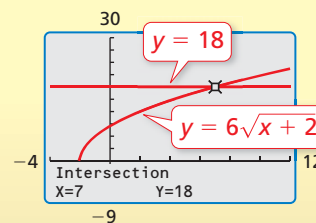
24. $5\sqrt{x} + 2 > 17$

25. $2\sqrt{x-8} < 24$

26. $7\sqrt[3]{x-3} \geq 21$

27. In a tsunami, the wave speeds (in meters per second) can be modeled by $s(d) = \sqrt{9.8d}$, where d is the depth (in meters) of the water. Estimate the depth of the water when the wave speed is 200 meters per second.

Check



5.5 Performing Function Operations (pp. 269–274)

Let $f(x) = 2x^{3/2}$ and $g(x) = x^{1/4}$. Find $\left(\frac{f}{g}\right)(x)$ and state the domain. Then evaluate the quotient when $x = 81$.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x^{3/2}}{x^{1/4}} = 2x^{(3/2-1/4)} = 2x^{5/4}$$

The functions f and g each have the same domain: all nonnegative real numbers. Because $g(0) = 0$, the domain of $\frac{f}{g}$ is restricted to all *positive* real numbers.

When $x = 81$, the value of the quotient is

$$\left(\frac{f}{g}\right)(81) = 2(81)^{5/4} = 2(81^{1/4})^5 = 2(3)^5 = 2(243) = 486.$$

28. Let $f(x) = 2\sqrt{3-x}$ and $g(x) = 4\sqrt[3]{3-x}$. Find $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$ and state the domain of each. Then evaluate $(fg)(2)$ and $\left(\frac{f}{g}\right)(2)$.
29. Let $f(x) = 3x^2 + 1$ and $g(x) = x + 4$. Find $(f+g)(x)$ and $(f-g)(x)$ and state the domain of each. Then evaluate $(f+g)(-5)$ and $(f-g)(-5)$.

5.6 Inverse of a Function (pp. 275–284)

Consider the function $f(x) = (x+5)^3$. Determine whether the inverse of f is a function. Then find the inverse.

Graph the function f . Notice that no horizontal line intersects the graph more than once. So, the inverse of f is a function. Find the inverse.

$$y = (x+5)^3 \quad \text{Set } y \text{ equal to } f(x).$$

$$x = (y+5)^3 \quad \text{Switch } x \text{ and } y.$$

$$\sqrt[3]{x} = y + 5 \quad \text{Take cube root of each side.}$$

$$\sqrt[3]{x} - 5 = y \quad \text{Subtract 5 from each side.}$$

► So, the inverse of f is $g(x) = \sqrt[3]{x} - 5$.

Find the inverse of the function. Then graph the function and its inverse.

30. $f(x) = -\frac{1}{2}x + 10$

31. $f(x) = x^2 + 8, x \geq 0$

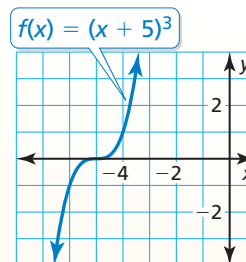
32. $f(x) = -x^3 - 9$

33. $f(x) = 3\sqrt{x} + 5$

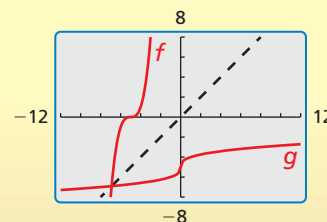
Determine whether the functions are inverse functions.

34. $f(x) = 4(x-11)^2, g(x) = \frac{1}{4}(x+11)^2$ 35. $f(x) = -2x + 6, g(x) = -\frac{1}{2}x + 3$

36. On a certain day, the function that gives U.S. dollars in terms of British pounds is $d = 1.587p$, where d represents U.S. dollars and p represents British pounds. Find the inverse function. Then find the number of British pounds equivalent to 100 U.S. dollars.



Check

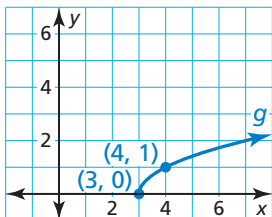


5 Chapter Test

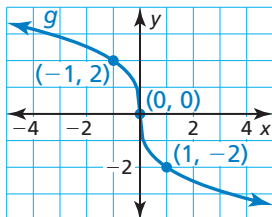
1. Solve the inequality $5\sqrt{x-3} - 2 \leq 13$ and the equation $5\sqrt{x-3} - 2 = 13$. Describe the similarities and differences in solving radical equations and radical inequalities.

Describe the transformation of f represented by g . Then write a rule for g .

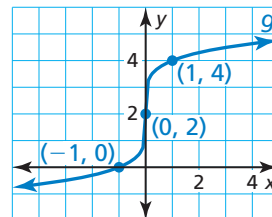
2. $f(x) = \sqrt{x}$



3. $f(x) = \sqrt[3]{x}$



4. $f(x) = \sqrt[5]{x}$



Simplify the expression. Explain your reasoning.

5. $64^{2/3}$

6. $(-27)^{5/3}$

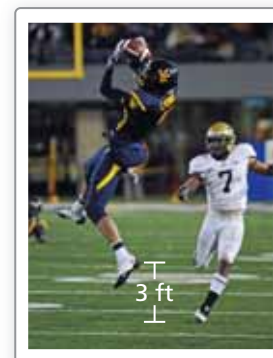
7. $\sqrt[4]{48xy^{11}z^3}$

8. $\frac{\sqrt[3]{256}}{\sqrt[3]{32}}$

9. Write two functions whose graphs are translations of the graph of $y = \sqrt{x}$. The first function should have a domain of $x \geq 4$. The second function should have a range of $y \geq -2$.
10. In bowling, a handicap is a change in score to adjust for differences in the abilities of players. You belong to a bowling league in which your handicap h is determined using the formula $h = 0.9(200 - a)$, where a is your average score. Find the inverse of the model. Then find the average for a bowler whose handicap is 36.
11. The basal metabolic rate of an animal is a measure of the amount of calories burned at rest for basic functioning. Kleiber's law states that an animal's basal metabolic rate R (in kilocalories per day) can be modeled by $R = 73.3w^{3/4}$, where w is the mass (in kilograms) of the animal. Find the basal metabolic rates of each animal in the table.

Animal	Mass (kilograms)
rabbit	2.5
sheep	50
human	70
lion	210

12. Let $f(x) = 6x^{3/5}$ and $g(x) = -x^{3/5}$. Find $(f + g)(x)$ and $(f - g)(x)$ and state the domain of each. Then evaluate $(f + g)(32)$ and $(f - g)(32)$.
13. Let $f(x) = \frac{1}{2}x^{3/4}$ and $g(x) = 8x$. Find $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$ and state the domain of each. Then evaluate $(fg)(16)$ and $\left(\frac{f}{g}\right)(16)$.
14. A football player jumps to catch a pass. The maximum height h (in feet) of the player above the ground is given by the function $h = \frac{1}{64}s^2$, where s is the initial speed (in feet per second) of the player. Find the inverse of the function. Use the inverse to find the initial speed of the player shown. Verify that the functions are inverse functions.

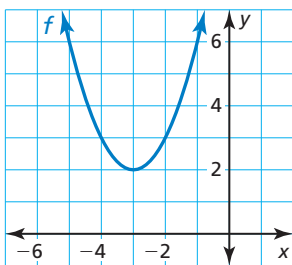


5 Cumulative Assessment

1. Identify three pairs of equivalent expressions. Assume all variables are positive. Justify your answer.

a	$a^{1/n}$	$\sqrt[n]{a^n}$	$a^{-1/n}$
$(\sqrt{a})^n$	$\sqrt{a^n}$	$\sqrt[n]{a}$	a^n

2. The graph represents the function $f(x) = (x - \square)^2 + \square$. Choose the correct values to complete the function.



-4	-3	-2	-1
1	2	3	4

3. In rowing, the boat speed s (in meters per second) can be modeled by $s = 4.62\sqrt[9]{n}$, where n is the number of rowers.
- Find the boat speeds for crews of 2 people, 4 people, and 8 people.
 - Does the boat speed double when the number of rowers doubles? Explain.
 - Find the time (in minutes) it takes each crew in part (a) to complete a 2000-meter race.
4. A polynomial function fits the data in the table. Use finite differences to find the degree of the function and complete the table. Explain your reasoning.

x	-4	-3	-2	-1	0	1	2	3
$f(x)$	28	2	-6	-2	8	18		

5. The area of the triangle is 42 square inches. Find the value of x .

