Chapter 8 Sequence & Series

8.1: Defining and Using Sequences & Series (pg. 410 – 413)

A **sequence** is an ordered list of numbers. A *finite sequence* is a function that has a limited number of terms and whose domain is the finite set $\{1, 2, 3, \ldots, n\}$. The values in the range are called the **terms** of the sequence.

Domain:	1	2	3	4	...	n	Relative position of each term
	↓	↓	↓	↓		↓	
Range:	a_1	a_2	a_3	a_4	...	a_n	Terms of the sequence

An *infinite sequence* is a function that continues without stopping and whose domain is the set of positive integers. Here are examples of a finite sequence and an infinite sequence.

Finite sequence: 2, 4, 6, 8 **Infinite sequence:** 2, 4, 6, 8, ...

A sequence can be specified by an equation, or *rule*. For example, both sequences above can be described by the rule $a_n = 2n$ or $f(n) = 2n$.

CONCEPT 1: WRITING THE TERMS OF SEQUENCES

1. Write the first six terms of (a) $a_n = 2n + 5$ and (b) $f(n) = (-3)^{n-1}$.

a) $a_n = 2n + 5$

$a_1 = 2(1) + 5 = 7$

$a_2 = 2(2) + 5 = 9$

$a_3 = 2(3) + 5 = 11$

$a_4 = 2(4) + 5 = 13$

$a_5 = 2(5) + 5 = 15$

$a_6 = 2(6) + 5 = 17$

b) $f(n) = (-3)^{n-1}$

$f(1) = (-3)^{1-1} = (-3)^0 = 1$

$f(2) = (-3)^{2-1} = (-3)^1 = -3$

$f(3) = (-3)^{3-1} = (-3)^2 = 9$

$f(4) = (-3)^{4-1} = (-3)^3 = -27$

$f(5) = (-3)^{5-1} = (-3)^4 = 81$

$f(6) = (-3)^{6-1} = (-3)^5 = -243$

2. Write the first six terms of (a) $a_n = \frac{1}{2}n - 3$ and (b) $f(n) = 2^{n-1}$.

a) $a_n = \frac{1}{2}n - 3$

$a_1 = \frac{1}{2}(1) - 3 = -2\frac{1}{2}$

$a_2 = \frac{1}{2}(2) - 3 = -2$

$a_3 = \frac{1}{2}(3) - 3 = -1\frac{1}{2}$

$a_4 = \frac{1}{2}(4) - 3 = -1$

$a_5 = \frac{1}{2}(5) - 3 = -\frac{1}{2}$

$a_6 = \frac{1}{2}(6) - 3 = 0$

b) $f(n) = 2^{n-1}$

$f(1) = 2^{1-1} = 2^0 = 1$

$f(2) = 2^{2-1} = 2^1 = 2$

$f(3) = 2^{3-1} = 2^2 = 4$

$f(4) = 2^{4-1} = 2^3 = 8$

$f(5) = 2^{5-1} = 2^4 = 16$

$f(6) = 2^{6-1} = 2^5 = 32$

****CONCEPT 2: WRITING RULES FOR SEQUENCES****

3. Describe the pattern, write the next term, and write a rule for the *n*th term of the sequences (a) $-1, -8, -27, -64, \ldots$ and (b) $0, 2, 6, 12, \ldots$.

a) $a_n = -n^3 = (-n)^3$

$a_5 = -5^3 = -125$

b) $a_n = n^2 - n = n(n-1)$

$a_5 = 5(5-1) = 5(4) = 20$

4. Describe the pattern, write the next term, and write a rule for the *n*th term of the sequences (a) $1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$ and (b) $0, 3, 8, 15, \ldots$.

a) $\frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \frac{6}{2}$

$a_n = \frac{n+1}{2}$

$a_5 = \frac{5+1}{2} = \frac{6}{2} = 3$

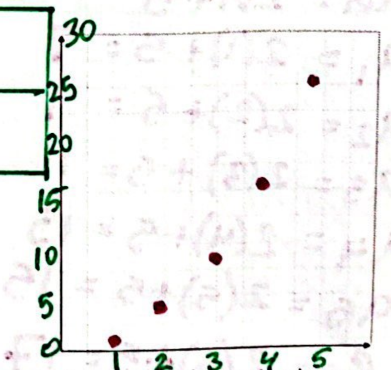
b) $a_n = n^2 - 1$

$a_5 = 5^2 - 1 = 25 - 1 = 24$

****CONCEPT 3: SOLVING A REAL-LIFE PROBLEM****

5. You work in a grocery store and are stacking apples in the shape of a square pyramid with 4 ~~seven~~ layers. Write a rule for the numbers of apples in each layer. Then graph the sequence.

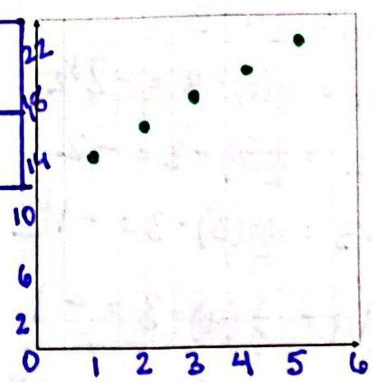
LAYERS	1	2	3	4	n
# OF APPLES	1	4	9	16	n^2



DON'T CONNECT THE POINTS!

6. In a video game, you start with 12 points and earn 2 more points for each minute that your avatar is active. Write a rule for the number of points for each minute that avatar is active. Then graph the sequence.

MINUTES	1	2	3	4	5	n
POINTS	14	16	18	20	22	$12 + 2n$



Series and Summation Notation

When the terms of a sequence are added together, the resulting expression is a series. A series can be finite or infinite.

Finite series: $2 + 4 + 6 + 8$

Infinite series: $2 + 4 + 6 + 8 + \cdots$

You can use summation notation to write a series. For example, the two series above can be written in summation notation as follows:

UP

Finite series: $2 + 4 + 6 + 8 = \sum_{i=1}^{4} 2i$ ← RULE

LOW (L.L.)

Infinite series: $2 + 4 + 6 + 8 + \cdots = \sum_{i=1}^{\infty} 2i$

For both series, the *index of summation* is i and the *lower limit of summation* is 1. The *upper limit of summation* is 4 for the finite series and ∞ (infinity) for the infinite series. Summation notation is also called sigma notation because it uses the uppercase Greek letter *sigma*, written Σ.

****CONCEPT 4: WRITING A SERIES USING SUMMATION NOTATION****

7. Write each series using summation notation.

a) $25 + 50 + 75 + \cdots + 250$

LL = i = 1

UP = 10

$a_i = 25i$

$\boxed{\sum_{i=1}^{10} 25i}$

b) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$

LL: i = 1

UP = ∞

$a_i = \frac{i}{i+1}$

$\boxed{\sum_{i=1}^{\infty} \frac{i}{i+1}}$

8. Write each series using summation notation.

a) $-1 + 2 + (-3) + 4 + (-5) + \cdots + 40$

LL = i = 1

UP = 40

$a_i = (-1)^i(i)$

$\boxed{\sum_{i=1}^{40} (-1)^i(i)}$

b) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

LL = i = 1

UP = ∞

$a_i = \frac{1}{i^2}$

$\boxed{\sum_{i=1}^{\infty} \frac{1}{i^2}}$

****CONCEPT 5: FINDING THE SUM OF A SERIES****

9. Find the sum.

$$\sum_{k=4}^{8} (3 + k^2) = (3+4^2) + (3+5^2) + (3+6^2) + (3+7^2) + (3+8^2)$$

$$19 + 28 + 39 + 52 + 67 = 205$$

10. Find the sum.

$$\sum_{k=3}^{7} (4k - 1) = [4(3)-1] + [4(4)-1] + [4(5)-1] + [4(6)-1] + [4(7)-1]$$

$$11 + 15 + 19 + 23 + 27 = 95$$

Formulas for Special Series

Sum of n terms of 1: $\sum_{i=1}^{n} 1 = n$

Sum of first n positive integers: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Sum of squares of first n positive integers: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

*****CONCEPT 6: USING A FORMULA*****

11. How many apples are in the stack in Example 5?

LL = i=1
UP = 4
$a_i = i^2$

$$\sum_{i=1}^{4} i^2 = \frac{4(4+1)(2(4)+1)}{6} = \frac{\cancel{4}^{2}(5)\cancel{(9)}^{3}}{\cancel{6}_{2}} = 30 \text{ APPLES}$$

12. As part of an experiment, a teacher gives one paper clip to the first student to enter the classroom, two to the second student, three to the third, and so on, until the 25th student receives 25 paper clips. What is the total number of paper clips the teacher gave to the students?

LL = i=1
UP = 25
$a_i = i$

$$\sum_{i=1}^{25} i = \frac{25(25+1)}{2} = \frac{25\cancel{(26)}^{13}}{\cancel{2}} = 325 \text{ PAPER CLIPS}$$

13.
$$\sum_{k=1}^{6} k = \frac{6(6+1)}{2} = \frac{\cancel{6}^{3}(7)}{\cancel{2}} = 21$$

14.
$$\sum_{i=1}^{34} 1 = 34$$

15.
$$\sum_{i=1}^{5} i^2 = \frac{5(5+1)(2(5)+1)}{6} = \frac{5\cancel{(6)}(11)}{\cancel{6}} = 55$$

16.
$$\sum_{k=3}^{7} \cancel{(k^2)-1}\ (k^2-1) = (3^2-1)+(4^2-1)+(5^2-1)+(6^2-1)+(7^2-1)$$
$$8 + 15 + 24 + 35 + 48 = 130$$

8.2: Analyzing Arithmetic Sequences & Series (pg. 418 – 421)

Arithmetic Sequence: The difference of consecutive terms is constant. Represents a *linear* function.

****CONCEPT 1: IDENTIFYING ARITHMETIC SEQUENCES****

(Ask yourself: "Are the consecutive differences constant?")

1. Tell whether each sequence is arithmetic.

a) −9, −2, 5, 12, 19, …

+7 +7 +7 +7 $d=7$

ARITHMETIC SEQ.

b) 23, 15, 9, 5, 3, …

−8 −6 −4

NOT AN ARITHMETIC SEQ.

2. Tell whether each sequence is arithmetic.

a) 1, 5, 10, 16, 23, …

4 5 6

NOT AN ARITHMETIC SEQ.

b) 4, 1, −2, −5, −8, …

−3 −3 −3 −3 $d=-3$

ARITHMETIC SEQ.

Rule for an Arithmetic Sequence

Algebra The nth term of an arithmetic sequence with first term a_1 and common difference d is given by:

$$a_n = a_1 + (n-1)d$$

****CONCEPT 2: WRITING A RULE FOR THE nTH TERM****

3. Write a rule for the nth term of each sequence. Then find a_{15}.

a) 3, 8, 13, 18, …

5 5 5 $d=5$ $a_1=3$

$a_n = a_1 + (n-1)d$

$a_n = 3 + (n-1)5$

$a_n = 3+5n-5$

$\boxed{a_n = 5n-2}$

$a_{15} = 5(15)-2$

$a_{15} = 75-2$

$\boxed{a_{15} = 73}$

b) 55, 47, 39, 31, …

−8 −8 −8 $d=-8$ $a_1=55$

$a_n = a_1 + (n-1)d$

$a_n = 55 + (n-1)(-8)$

$a_n = 55-8n+8$

$\boxed{a_n = -8n+63}$

$a_{15} = -8(15)+$

$a_{15} = -120+$

$\boxed{a_{15} = -57}$

4. Write a rule for the nth term of each sequence. Then find a_{15}.

a) 7, 10, 13, 16, …

3 3 3 $d=3$ $a_1=7$

$a_n = 7+(n-1)3$

$a_n = 7+3n-3$

$\boxed{a_n = 3n+4}$

$a_{15} = 3(15)+4$

$\boxed{a_{15} = 49}$

b) 15, 8, 1, −6, …

−7 −7 −7 $d=-7$ $a_1=15$

$a_n = 15+(n-1)(-7)$

$a_n = 15-7n+7$

$\boxed{a_n = -7n+22}$

$a_{15} = -7(15)+22$

$a_{15} = -105+22$

$\boxed{a_{15} = -83}$

****CONCEPT 3: WRITING A RULE GIVEN A TERM AND COMMON DIFFERENCE****

5. One term of an arithmetic sequence is $a_{19} = -45$. The common difference is $d = -3$. Write a rule for the nth term. Then graph the first six terms of the sequence.

$a_n = a_1 + (n-1)d$

$a_{19} = a_1 + (19-1)d$

$-45 = a_1 + (18)(-3)$

$-45 = a_1 - 54$

$+54 \quad +54$

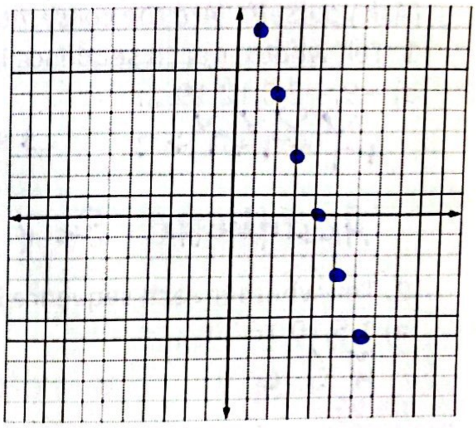
$a_1 = 9$

$a_n = 9 + (n-1)(-3)$

$a_n = 9 - 3n + 3$

$\boxed{a_n = -3n + 12}$

n	a_n
1	9
2	6
3	3
4	0
5	-3
6	-6



6. One term of an arithmetic sequence is $a_{12} = 52$. The common difference is $d = 6$. Write a rule for the nth term. Then graph the first ~~six~~ 5 terms of the sequence.

$a_{12} = a_1 + (12-1)d$

$52 = a_1 + (11)(6)$

$52 = a_1 + 66$

$-66 \quad -66$

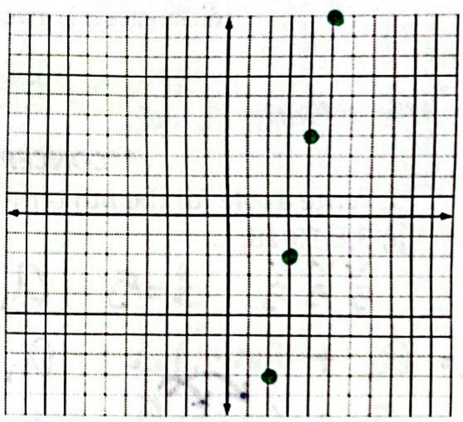
$a_1 = -14$

$a_n = -14 + (n-1)6$

$a_n = -14 + 6n - 6$

$\boxed{a_n = 6n - 20}$

n	a_n
1	-14
2	-8
3	-2
4	4
5	10



****CONCEPT 4: WRITING A RULE GIVEN TWO TERMS****

7. Two terms of an arithmetic sequence are $a_7 = 17$ and $a_{26} = 93$. Write a rule for the nth term.

$a_{26} = a_1 + (26-1)d = 93 \Rightarrow \cancel{a_1} + 25d = 93$

$a_7 = a_1 + (7-1)d = 17 \Rightarrow -\cancel{a_1} + -6d = -17$

$19d = 76$

$\frac{19d}{19} = \frac{76}{19}$

$d = 4$

$a_1 + 25(4) = 93$

$a_1 + 100 = 93$

$-100 \quad -100$

$a_1 = -7$

$a_n = -7 + (n-1)4$

$a_n = -7 + 4n - 4$

$\boxed{a_n = 4n - 11}$

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8. Two terms of an arithmetic sequence are $a_5 = -43$ and $a_{12} = -8$. Write a rule for the nth term.

$a_{12} = a_1 + (12-1)d = -8 \Rightarrow \quad a_1 + 11d = -8 \qquad a_1 + 11(5) = -8$

$a_5 = a_1 + (5-1)d = -43 \Rightarrow \quad -a_1 + -4d = +43 \qquad a_1 + 55 = -8$

$-55 \quad -55$

$\frac{7d}{7} = \frac{35}{7} \qquad a_1 = -63$

$d = 5$

$a_n = -63 + (n-1)5$

$a_n = -63 + 5n - 5$

$\boxed{a_n = 5n - 68}$

The Sum of a Finite Arithmetic Series

The sum of the first n terms of an arithmetic series is $S_n = n\left(\frac{a_1 + a_n}{2}\right)$.

In words, S_n is the mean of the first and nth terms, multiplied by the number of terms.

*****CONCEPT 5: FINDING THE SUM OF A FINITE SERIES*****

9. Find the sum.

$$\sum_{i=1}^{20}(3i+7)$$

$S_{20} = \cancel{20}^{10}\left(\frac{10+67}{\cancel{2}}\right) = 10(77) = 770$

$a_1 = 3(1)+7 = 10$

$\boxed{S_{20} = 770}$

$a_{20} = 3(20)+7 = 67$

10. Find the sum.

$$\sum_{k=1}^{18}(6k-2)$$

$S_{18} = \cancel{18}^{9}\left(\frac{4+106}{\cancel{2}}\right) = 9(110) = 990$

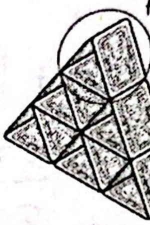
$a_1 = 6(1)-2 = 4$

$\boxed{S_{18} = 990}$

$a_{18} = 6(18)-2 = 108-2 = 106$

CONCEPT 6: SOLVING A REAL-LIFE PROBLEM

11. You are making a house of cards similar to the one shown.
a) Write a rule for the number of cards in the *n*th row when the top row is 1.
b) How many cards do you need to make a house of cards with 12 rows?



a) $a_1 = 3 \quad d = 3$

$a_n = a_1 + (n-1)d$

$a_n = 3 + (n-1)3$

$a_n = 3 + 3n - 3$

$\boxed{a_n = 3n}$

b) $a_1 = 3$

$a_{12} = 3(12) = 36$

$S_{12} = \cancel{12}^{6}\left(\frac{3+36}{\cancel{2}}\right) = 6(39)$

$\boxed{S_{12} = 234 \text{ CARDS}}$

12. A camp counselor earns $300 for the first week of summer camp, $350 of the second week, $400 for the third, and so on for up to 8 weeks of the summer.
a) Write a rule for the amount the counselor earns for the *n*th week of summer.
b) How much does the counselor earn for 7 weeks of summer camp?

a) $a_1 = 300 \quad d = 50$

$a_n = 300 + (n-1)50$

$a_n = 300 + 50n - 50$

$\boxed{a_n = 50n + 250}$

b) $a_1 = 300$

$a_7 = 50(7) + 250 = 350 + 250 = 600$

$S_7 = 7\left(\frac{300+600}{2}\right) = 7\left(\frac{900}{2}\right)$

$\boxed{S_7 = 7(450) = \$3,150}$

8.3: Geometric Sequences & Series (pg. 426 – 429)

****CONCEPT 1: IDENTIFYING GEOMETRIC SEQUENCES*****

1. Tell whether each sequence is geometric.

COMMON RATIO : r

a) 6,12,20,30,42, ...

$\frac{a_2}{a_1} = \frac{12}{6} = 2,\ \frac{a_3}{a_2} = \frac{20}{12} = \frac{5}{3},\ \frac{a_4}{a_3} = \frac{30}{20} = \frac{2}{3}$

NOT A GEOMETRIC SEQ.

b) 256,64,16,4,1, ...

$\frac{64}{256} = \frac{1}{4},\ \frac{16}{64} = \frac{1}{4},\ \frac{4}{16} = \frac{1}{4}$

$r = 1/4$ GEOMETRIC SEQ.

2. Tell whether each sequence is geometric.

a) $-3, -18, -108, -648, -3888, \ldots$

$\frac{-18}{-3} = 6,\ \frac{-108}{-18} = 6,\ \frac{-648}{-108} = 6$

$r = 6$ GEOMETRIC SEQ.

b) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}, \ldots$

$\frac{\frac{3}{4}}{\frac{1}{2}} = \frac{3}{\cancel{4}_2} \cdot \frac{\cancel{2}}{1} = \frac{3}{2},\ \frac{\frac{5}{6}}{\frac{3}{4}} = \frac{5}{\cancel{6}_3} \cdot \frac{\cancel{4}^2}{3} = \frac{10}{9}$

NOT A GEOMETRIC SEQ

Rule for a Geometric Sequence

Algebra The nth term of a geometric sequence with first term a_1 and common ratio r is given by:

$$a_n = a_1 r^{n-1}$$

****CONCEPT 2: WRITING A RULE FOR THE nTH TERM*****

3. Write a rule for the nth term of each sequence. Then find a_8.

a) 5,15,45,135, ...

$a_1 = 5 \quad r = 3$

$a_n = a_1 r^{n-1}$

$\boxed{a_n = 5(3)^{n-1}}$

$\boxed{a_8 = 5(3)^{8-1} = 5(3)^7 = 5(2187) = 10935}$

b) $88, -44, 22, -11, \ldots$

$a_1 = 88 \quad r = -\frac{1}{2}$

$\boxed{a_n = 88\left(-\frac{1}{2}\right)^{n-1}}$

$a_8 = 88\left(-\frac{1}{2}\right)^{8-1} = 88\left(-\frac{1}{2}\right)^7$

$\boxed{a_8 = \frac{-88}{128} = \frac{-11}{16} = -0.687}$

4. Write a rule for the nth term of each sequence. Then find a_8.

a) $6, 2, \frac{2}{3}, \frac{2}{9}, \ldots$

$a_1 = 6 \quad r = \frac{1}{3}$

$\boxed{a_n = 6\left(\frac{1}{3}\right)^{n-1}}$

$a_8 = 6\left(\frac{1}{3}\right)^{8-1} = 6\left(\frac{1}{3}\right)^7 = \frac{6}{2187}$

$\boxed{a_8 = \frac{2}{729} = 0.003}$

b) 10,20,40,80, ...

$a_1 = 10 \quad r = 2$

$\boxed{a_n = 10(2)^{n-1}}$

$a_8 = 10(2)^{8-1} = 10(2)^7 = 10(128)$

$\boxed{a_8 = 1280}$

****CONCEPT 3: WRITING A RULE GIVEN A TERM AN A COMMON RATIO****

5. One term of a geometric sequence is $a_4 = 12$. The common ration is $r = 2$. Write a rule for the nth term. Then graph the first six terms of the sequence.

$a_n = a_1(r)^{n-1}$

$a_4 = a_1(2)^{4-1} = 12$

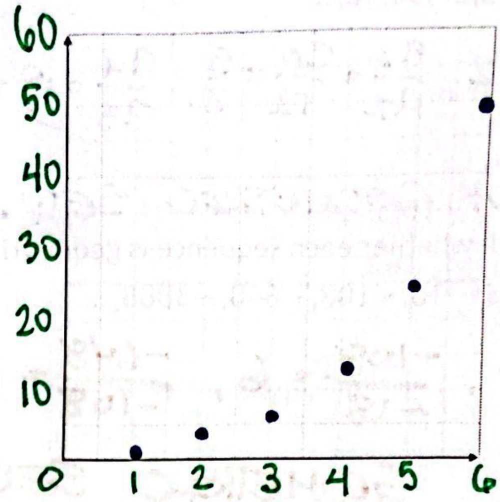
$a_1(2)^3 = 12$

$\frac{8a_1}{8} = \frac{12}{8}$

$a_1 = 3/2$

$\boxed{a_n = \left(\frac{3}{2}\right)(2)^{n-1}}$

n	a_n
1	3/2
2	3
3	6
4	12
5	24
6	48



6. One term of a geometric sequence is $a_5 = 32$. The common ration is $r = \frac{2}{3}$. Write a rule for the nth term. Then graph the first six terms of the sequence.

$a_5 = a_1\left(\frac{2}{3}\right)^{5-1} = 32$

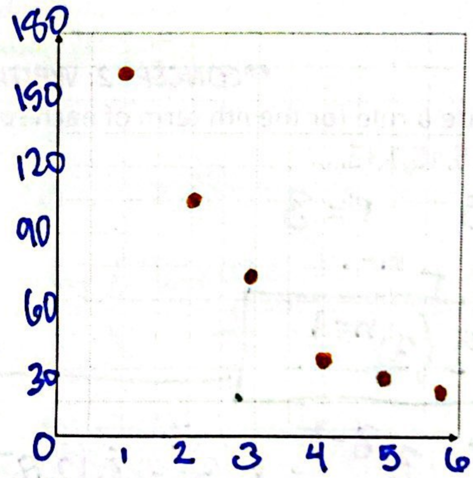
$a_1\left(\frac{2}{3}\right)^4 = 32$

$\cancel{\left(\frac{81}{16}\right)}\cancel{\frac{16}{81}}a_1 = \cancel{32}^2\left(\frac{81}{\cancel{16}}\right)$

$a_1 = 162$

$\boxed{a_n = 162\left(\frac{2}{3}\right)^{n-1}}$

n	a_n
1	162
2	108
3	72
4	48
5	32
6	$21.\overline{3}$



****CONCEPT 4: WRITING A RULE GIVEN TWO TERMS****

7. Two terms of a geometric sequence are $a_2 = 12$ and $a_5 = -768$. Write a rule for the nth term.

$a_2 = a_1(r)^{2-1} = 12$

$\frac{a_1 r}{r} = \frac{12}{r}$

$a_1 = \frac{12}{r} = \frac{12}{-4} = -3$

$a_5 = \left(\frac{12}{r}\right)r^{5-1} = -768$

$\frac{12}{r} \cdot r^4 = -768$

$\frac{12r^3}{12} = \frac{-768}{12}$

$\sqrt[3]{r^3} = \sqrt[3]{-64}$

$r = -4$

$\boxed{a_n = (-3)(-4)^{n-1}}$

8. Two terms of a geometric sequence are $a_3 = 20$ and $a_6 = -160$. Write a rule for the nth term.

$a_3 = a_1 r^{3-1} = 20$

$\frac{a_1 r^2}{r^2} = \frac{20}{r^2}$

$a_1 = \frac{20}{r^2} = \frac{20}{(-2)^2} = \frac{20}{4} = 5$

$a_6 = \left(\frac{20}{r^2}\right) r^{6-1} = -160$

$\frac{20}{r^2} r^5 = -160$

$\frac{20r^3}{20} = \frac{-160}{20}$

$\sqrt[3]{r^3} = \sqrt[3]{-8}$

$r = -2$

$\boxed{a_n = 5(-2)^{n-1}}$

The Sum of a Finite Geometric Series

The sum of the first n terms of a geometric series with common ratio $r \neq 1$ is

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right).$$

*****CONCEPT 5: FINDING THE SUM OF A GEOMETRIC SERIES*****

9. Find the sum.

$$\sum_{k=1}^{10} 4(3)^{k-1}$$

$S_{10} = 4\left(\frac{1-3^{10}}{1-3}\right) = \cancel{4}^{2}\left(\frac{+59048}{+\cancel{2}}\right) = 118096$

$a_1 = 4(3)^{1-1} = 4(3)^0 = 4$

$r = 3$

$n = 10$

$\boxed{S_{10} = 118096}$

10. Find the sum.

$$\sum_{k=1}^{6} \frac{1}{2}(5)^k$$

$S_6 = \frac{5}{2}\left(\frac{1-5^6}{1-5}\right) = \frac{5}{2}\left(\frac{+15624}{+4}\right) = \frac{78120}{8}$

$a_1 = \frac{1}{2}(5)^1 = \frac{5}{2}$

$r = 5$

$n = 6$

$\boxed{S_6 = 9765}$