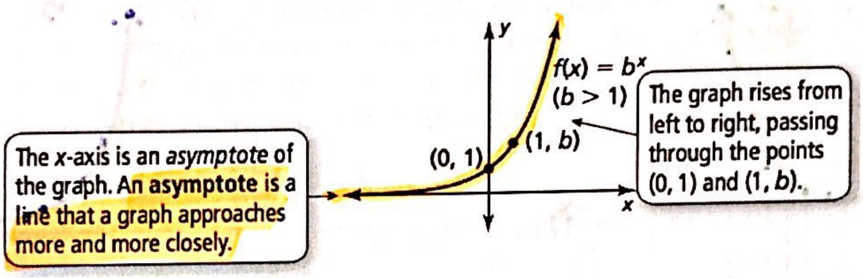
Chapter 6 Exponential and Logarithmic Functions

6.1: Exponential Growth & Decay Functions (pg. 296 – 299)

exponential function has the form $y = ab^x$.

Parent Function for Exponential Growth Functions

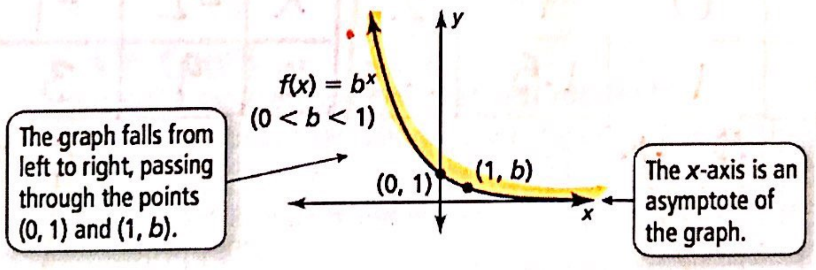
The function $f(x) = b^x$, where $b > 1$, is the parent function for the family of exponential growth functions with base b. The graph shows the general shape of an exponential growth function.



The domain of $f(x) = b^x$ is all real numbers. The range is $y > 0$.

Parent Function for Exponential Decay Functions

The function $f(x) = b^x$, where $0 < b < 1$, is the parent function for the family of exponential decay functions with base b. The graph shows the general shape of an exponential decay function.



The domain of $f(x) = b^x$ is all real numbers. The range is $y > 0$.

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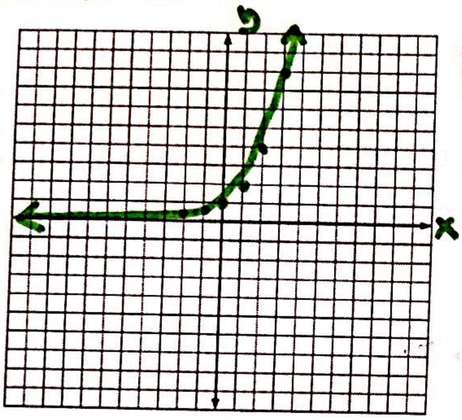
CONCEPT 1: GRAPHING EXPONENTIAL GROWTH & DECAY FUNCTIONS

1. Tell whether each function represents exponential growth (b > 1) or decay (0 < b < 1). Then graph the function.

a) $y = (2)^x$

b GROWTH

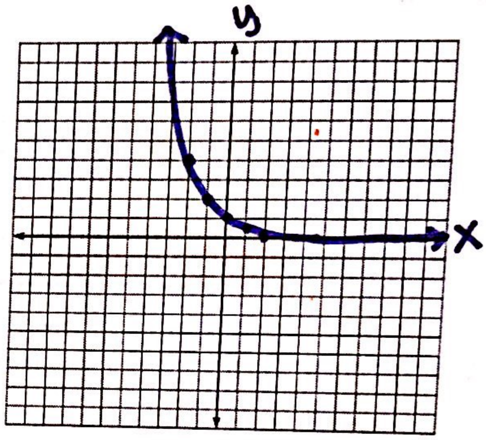
x	-2	-1	0	1	2
y	1/4	1/2	1	2	4



b) $y = \left(\frac{1}{2}\right)^x$

b DECAY

x	-2	-1	0	1	2
y	4	2	1	1/2	1/4

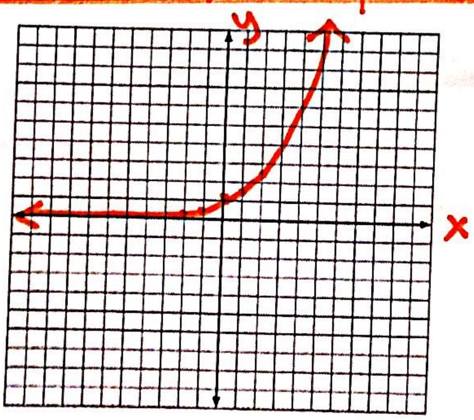


2. Tell whether each function represents exponential growth (b > 1) or decay (0 < b < 1). Then graph the function.

a) $y = (1.5)^x$

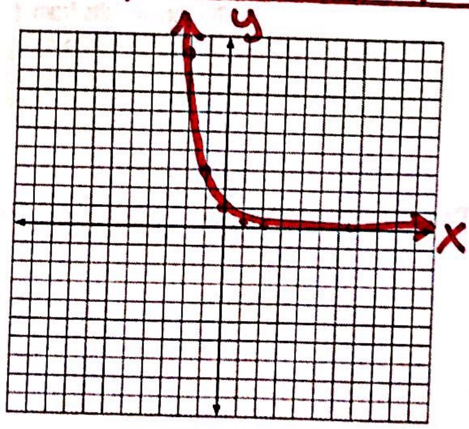
b GROWTH

x	-2	-1	0	1	2
y	$0.\overline{4}$	$0.\overline{6}$	1	1.5	2.25



b) $y = \left(\frac{1}{3}\right)^x$

x	-2	-1	0	1	2
y	9	3	1	1/3	1/9



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Exponential Models

Some real-life quantities increase or decrease by a fixed percent each year (or some other time period). The amount y of such a quantity after t years can be modeled by one of these equations.

Exponential Growth Model

$$y = a(1 + r)^t$$

Exponential Decay Model

$$y = a(1 - r)^t$$

Note that a is the initial amount and r is the percent increase or decrease written as a decimal. The quantity $1 + r$ is the growth factor, and $1 - r$ is the decay factor.

*****CONCEPT 2: SOLVING A REAL-LIFE PROBLEM*****

3. The value of a car y (in thousands of dollars) can be approximated by the model $y = 25(0.85)^t$, where t is the number of years since the car was new.
a) Tell whether the model represents exponential growth or exponential decay.

$b = 0.85 \quad 0 < 0.85 < 1$ DECAY

b) Identify the annual percent increase or decrease in the value of the car.

$1 - r = b$
$1 - r = 0.85$
$-1 \quad -1$

$\frac{-r}{-1} = \frac{-0.15}{-1}$

$r = 0.15 = 15\%$ DECREASE

$\boxed{b > 1;\ \text{GROWTH} \quad 1 + r = b}$

4. The value of a car y (in thousands of dollars) can be approximated by the model $y = 31(0.92)^t$, where t is the number of years since the car was new.
a) Tell whether the model represents exponential growth or exponential decay.

$b = 0.92 \quad 0 < 0.92 < 1$ DECAY

b) Identify the annual percent increase or decrease in the value of the car.

$1 - r = b$
$-1 \quad -1$
$-r = 0.92 - 1$

$\frac{-r}{-1} = \frac{-0.08}{-1}$
$r = 0.08$

$r = 0.08 = 8\%$ DECREASE

*****CONCEPT 3: WRITING AN EXPONENTIAL MODEL*****

5. In 2000, the world population was about 6.09 billion. During the next 13 years, the world population increase by about 1.18% each year.
a) Write an exponential growth model giving the population y (in billions) t years after 2000.
b) Estimate the world population in 2005.

a) $y = a(1+r)^t = y = 6.09(1 + 0.0118)^t = \boxed{y = 6.09(1.0118)^t}$

b) $t \Rightarrow 2005 \Rightarrow t = 5$

$y = 6.09(1.0118)^5 = 6.09(1.06) = \boxed{6.49 \text{ BILLION PEOPLE}}$

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DECAY MODEL $b = 0.5$

CONCEPT 4: REWRITING AN EXPONENTIAL FUNCTION

6. The amount y (in grams) of the radioactive isotope chromium-51 remaining after *t* days is $y = a(0.5)^{\frac{t}{28}}$, where *a* is the initial amount (in grams). What percent of the chromium-51 decays each day?

$y = a(0.5)^{t/28}$

$y = a\left[(0.5)^{1/28}\right]^t \Rightarrow y = a\left[0.9755\right]^t$

$1 - r = b$

$1 - r = 0.9755$

$-1 \quad -1$

$-r = -0.0245$

$r = 0.0245 = 2.$

DECREA

7. The amount y (in grams) of the radioactive isotope barium-140 remaining after *t* days is $y = a(0.5)^{\frac{t}{13}}$, where *a* is the initial amount (in grams). What percent of the barium-140 decays each day?

$y = a(0.5)^{t/13}$

$y = a\left[(0.5)^{1/13}\right]^t = a\left[0.9481\right]^t$

$1 - r = 0.9481$

$-1 \quad -1$

$-r = -0.0519$

$r = 0.0519 = 5.19\%$ DECREASE

Compound Interest

Consider an initial principal *P* deposited in an account that pays interest at an annual rate *r* (expressed as a decimal), compounded *n* times per year. The amount *A* in the account after *t* years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

CONCEPT 5: FINDING THE BALANCE IN AN ACCOUNT

8. You deposit \$9000 in an account that pays 1.46% annual interest. Find the balance after 3 years when the interest is compounded quarterly.

$A = 9000\left(1 + \frac{0.0146}{4}\right)^{4 \cdot 3}$

$A = 9000(1.00365)^{12} = \$9402.\underline{21}$

9. You deposit \$8600 in an account that pays 1.32% annual interest. Find the balance after 4 years when the interest is compounded quarterly.

$A = 8600\left(1 + \frac{0.0132}{4}\right)^{4 \cdot 4}$

$A = 8600(1.0033)^{16} = \$9065.\underline{49}$

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6.2: The Natural Base *e* (pg. 304 – 306)

The Natural Base e

The natural base e is irrational. It is defined as follows:

As x approaches $+\infty$, $\left(1 + \frac{1}{x}\right)^x$ approaches $e \approx 2.71828182846$.

*****CONCEPT 1: SIMPLIFYING NATURAL BASE EXPRESSIONS*****

1. Simplifying each expression.

a) $e^3 \cdot e^6$

$e^{3+6} = e^9 \approx 8103.1$

b) $\frac{16e^5}{4e^4}$

$4e = 4e^1 \approx 10.9$

c) $(3e^{-4x})^2$

$3^2 e^{-8x} = \frac{9}{e^{8x}}$

2. Simplifying each expression.

a) $e^2 \cdot e^9$

$e^{11} \approx 59874.1$

b) $\frac{25e^{13}}{5e^{12}}$

$5e \approx 13.6$

c) $(2e^{-3x})^5$

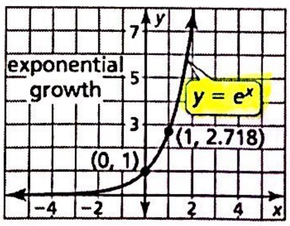
$2^5 e^{-15x} \Rightarrow \frac{32}{e^{15x}}$

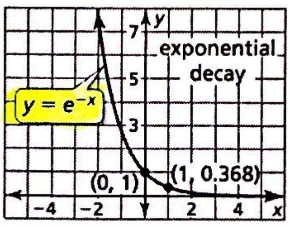
Natural Base Functions

A function of the form $y = ae^{rx}$ is called a *natural base exponential function.*

- When $a > 0$ and $r > 0$, the function is an exponential growth function.
- When $a > 0$ and $r < 0$, the function is an exponential decay function.

The graphs of the basic functions $y = e^x$ and $y = e^{-x}$ are shown.





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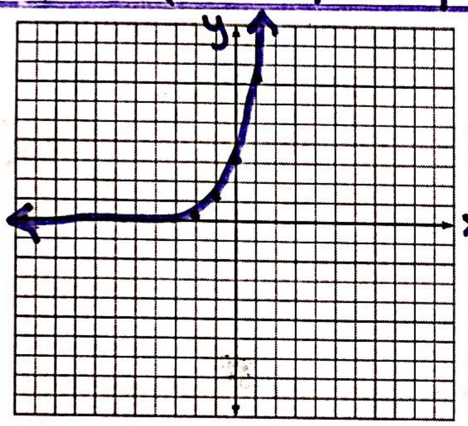
CONCEPT 2: GRAPHING NATURAL BASE FUNCTIONS

3. Tell whether each function represents exponential growth or exponential decay. Then graph the function.

a) $y = 3e^{x}$

a = 3 r = 1 GROWTH

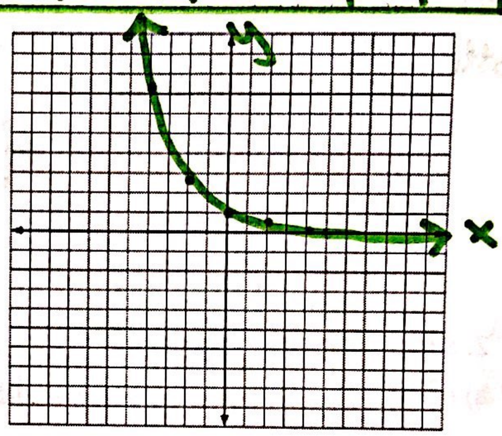
x	-2	-1	0	1	2
y	.4	1.1	3	8.2	22.2



b) $f(x) = e^{-0.5x}$

a = 1 r = -0.5 DECAY

x	-4	-2	0	2	4
y	7.4	2.7	1	.4	.1

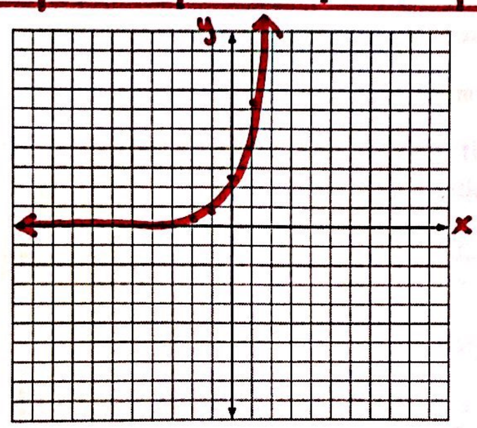


4. Tell whether each function represents exponential growth or exponential decay. Then graph the function.

a) $f(x) = 2.5e^{x}$

a = 2.5 r = 1 GROWTH

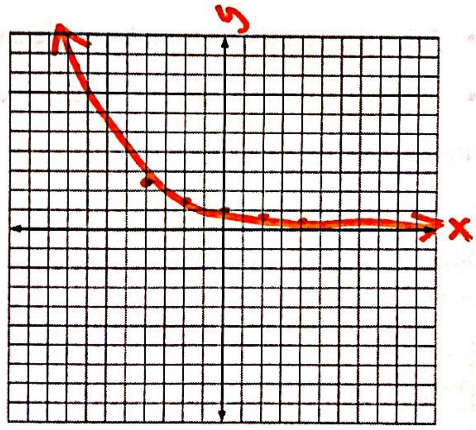
x	-2	-1	0	1	2
y	.33	.92	2.5	6.3	18.5



b) $y = e^{-0.2x}$

a = 1 r = -0.2 DECAY

x	-4	-2	0	2	4
y	2.2	1.5	1	0.7	0.4



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Continuously Compounded Interest

When interest is compounded *continuously*, the amount A in an account after t years is given by the formula

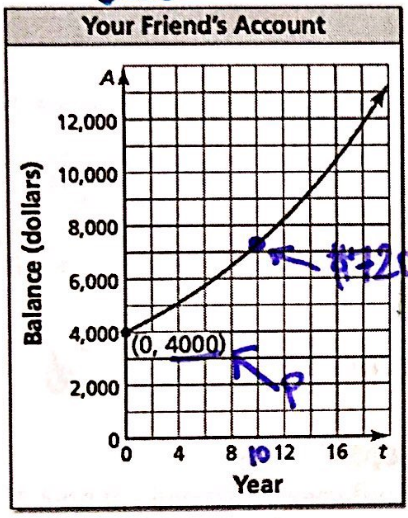
$$A = Pe^{rt}$$

where P is the principal and r is the annual interest rate expressed as a decimal.

CONCEPT 3: MODELING WITH MATHEMATICS

5. You and your friend each have accounts that earn annual interest compounded continuously. The balance A (in dollars) of your account after t years can be modeled by $A = 4500e^{0.04t}$. The graph shows the balance of your friend's account over time. Which account has a greater principal? Which has a greater balance after 10 years? YOUR FRIEND

↑ you.



$t=10$

$A = 4500e^{0.04t} = 4500e^{0.4} \approx \$6713.21$

P

6. You deposit \$4250 in an account that earns 5% annual interest compounded continuously. Compare the balance after 10 years with the account in Example ~~6~~ 5

$A = 4250e^{.05(10)} = 4250e^{0.5} \approx \$7007.07$

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6.3 Logarithms & Logarithmic Functions (pg. 310 – 313)

Definition of Logarithm with Base b

Let b and y be positive real numbers with $b \neq 1$. The **logarithm of y with base b** is denoted by $\log_b y$ and is defined as

$$\log_b y = x \quad \text{if and only if} \quad b^x = y.$$

The expression $\log_b y$ is read as "log base b of y."

*****CONCEPT 1: REWRITING LOGARITHMIC FUNCTIONS*****

1. Rewrite each equation in exponential form.

a) $\log_2 16 = 4$ b) $\log_4 1 = 0$ c) $\log_{12} 12 = 1$ d) $\log_{\frac{1}{4}} 4 = -1$

Logarithmic Form	Exponential Form
a) $\log_2 16 = 4$	$2^4 = 16$
b) $\log_4 1 = 0$	$4^0 = 1$
c) $\log_{12} 12 = 1$	$12^1 = 12$
d) $\log_{1/4} 4 = -1$	$1/4^{-1} = 4$ $\Rightarrow \frac{1}{1/4} = \frac{1}{1} \cdot \frac{4}{1} = 4$

*****CONCEPT 2: REWRITING EXPONENTIAL FUNCTIONS*****

2. Rewrite each equation in logarithmic form.

a) $5^2 = 25$ b) $10^{-1} = 0.1$ c) $8^{\frac{2}{3}} = 4$ d) $6^{-3} = \frac{1}{216}$

Exponential Form	Logarithmic Form
a) $5^2 = 25$	$\log_5 25 = 2$
b) $10^{-1} = 0.1$	$\log_{10} 0.1 = -1$
c) $8^{2/3} = 4$	$\log_8 4 = 2/3$
d) $6^{-3} = \frac{1}{216}$	$\log_6 \frac{1}{216} = -3$

*****CONCEPT 3: EVALUATING LOGARITHMIC EXPRESSIONS*****

HINT: To help you find the value of log(b)y, ask which power of b gives you y

3. Evaluate each logarithm.

a) $\log_4 64$ b) $\log_5 0.2$ c) $\log_{\frac{1}{5}} 125$ d) $\log_{36} 6$

a) $\log_4 64 \to 4^? = 64 \to 4^3 = 64$ $\log_4 64 = 3$

b) $\log_5 0.2 \to 5^? = 0.2 \to 5^{-1} = 0.2$ $\log_5 0.2 = -1$

c) $\log_{1/5} 125 \to \frac{1}{5}^? = 125 \to \frac{1}{5}^{-3} = 125$ $\log_{1/5} 125 = -3$

d) $\log_{36} 6 \to 36^? = 6 = 36^{1/2} = 6$ $\log_{36} 6 = 1/2$

$\to \sqrt{36}$

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* NO BASE GIVEN MEANS BASE IS 10

****CONCEPT 4: EVALUATING COMMON & NATURAL LOGARITHMS****

4. Evaluate (a) log 8 and (b) ln 0.3 using a calculator. Round your answer to the third decimal place.

a) $\log 8 \Rightarrow \log_{10} 8 \approx 0.903$ b) $\ln 0.3 = -1.204$

Using Inverse Properties

By the definition of a logarithm, it follows that the logarithmic function $g(x) = \log_b x$ is the inverse of the exponential function $f(x) = b^x$. This means that

$$g(f(x)) = \log_b b^x = x \quad \text{and} \quad f(g(x)) = b^{\log_b x} = x.$$

In other words, exponential functions and logarithmic functions "undo" each other.

****CONCEPT 5: USING INVERSE PROPERTIES****

5. Simplify (a) $10^{\log 4}$ and (b) $\log_5 25^x$

a) $\cancel{10^{\log_{10}}}\,4 = 4$ b) $\cancel{\log_5}(5^2)^x = 2x$

6. Simplify (a) $10^{\log 7}$ and (b) $\log_3 27^x$

a) $\cancel{10^{\log_{10}}}\,7 = 7$ b) $\cancel{\log_3}(3^3)^x = 3x$

****CONCEPT 6: FINDING INVERSE FUNCTIONS****

* THE INVERSE OF ln IS e.

7. Find the inverse of each function

a) $f(x) = 6^x$

$g(x) = \log_6 x$

b) $y = \ln(x + 3)$

$x = \ln(y+3)$

$e^x = \cancel{e^{\ln}}(y+3)$

$e^x - 3 = y + 3 - 3$

$y = e^x - 3$

8. Find the inverse of the function.

a) $f(x) = 11^x$

$g(x) = \log_{11} x$

b) $y = \ln(x + 6)$

$x = \ln(y+6)$

$e^x = \cancel{e^{\ln}}(y+6)$

$e^x - 6 = y + \cancel{6} - \cancel{6}$

$y = e^x - 6$

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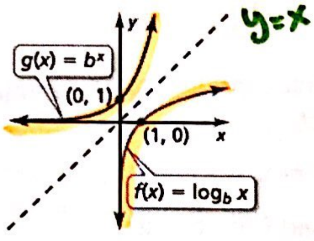
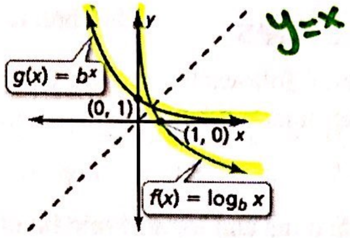
Parent Graphs for Logarithmic Functions

The graph of $f(x) = \log_b x$ is shown below for $b > 1$ and for $0 < b < 1$.
Because $f(x) = \log_b x$ and $g(x) = b^x$ are inverse functions, the graph of $f(x) = \log_b x$ is the reflection of the graph of $g(x) = b^x$ in the line $y = x$.

Graph of $f(x) = \log_b x$ for $b > 1$ Graph of $f(x) = \log_b x$ for $0 < b < 1$





Note that the y-axis is a vertical asymptote of the graph of $f(x) = \log_b x$. The domain of $f(x) = \log_b x$ is $x > 0$, and the range is all real numbers.

****CONCEPT 7: GRAPHING A LOGARITHMIC FUNCTION****

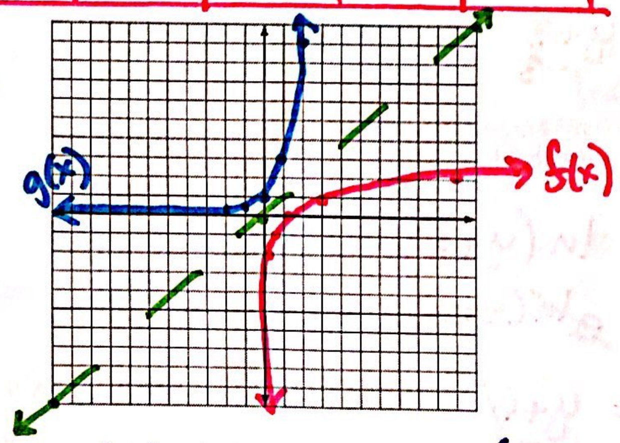
9. a) Graph $f(x) = \log_3 x$

$g(x) = 3^x$

x	-2	-1	0	1	2
y	1/9	1/3	1	3	9

$f(x) = \log_3 x$

x	1/9	1/3	1	3	9
y	-2	-1	0	1	2



ASYMPTOTE: $x=0$ (y-axis)
DOMAIN: $x>0$
RANGE: $\mathbb{R}$ (ALL REAL NUMBERS)

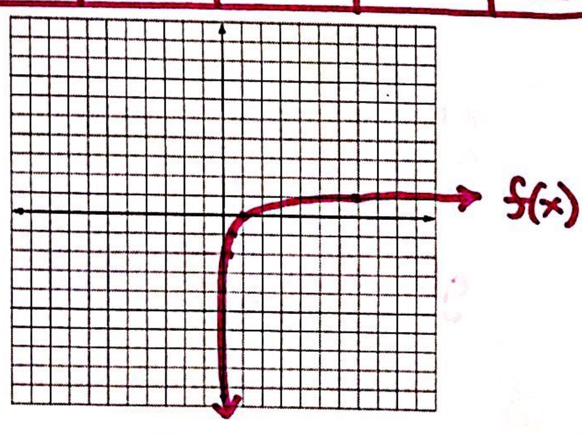
b) Graph $y = \log_7 x$

$g(x) = 7^x$

x	-2	-1	0	1	2
y	1/49	1/7	1	7	49

$f(x) = \log_7 x$

x	1/49	1/7	1	7	49
y	-2	-1	0	1	2



A: $x=0$
D: $x>0$
R: $\mathbb{R}$

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