4.5 Solving Polynomial Equations (pg. 190-193)

- Repeated Solution: A solution that appears more than once

*****CONCEPT 1: SOLVING A POLYNOMIAL BY FACTORING*****

1. Solve $2x^3 - 12x^2 + 18x = 0$

$2x(x^2 - 6x + 9) = 0$

$2x(x-3)(x-3) = 0$

$\frac{2x}{2} = \frac{0}{2}$ $\quad x - 3 = 0$, $+3$ $+3$ $\quad x - 3 = 0$, $+3$ $+3$

$\boxed{x = 0, 3, 3}$

*****CONCEPT 2: FINDING ZEROS OF A POLYNOMIAL*****

2. Find the zeros of $f(x) = -2x^4 + 16x^2 - 32$. Then sketch a graph of the function.

$-2x^4 + 16x^2 - 32 = 0$ $\quad (0, c) \rightarrow (0, -32)$

$-2(x^4 - 8x^2 + 16) = 0$

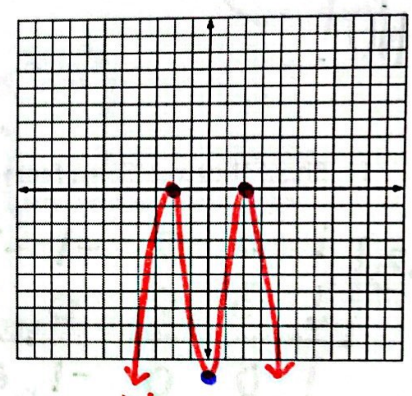
$-2(x^2 - 4)(x^2 - 4) = 0$

$-2(x+2)(x-2)(x-2)(x+2) = 0$

$\cancel{-2 \neq 0}$ $\quad x + 2 = 0$, -2 -2 $\quad x - 2 = 0$, $+2$ $+2$ $\quad x - 2 = 0$, $+2$ $+2$ $\quad x + 2 = 0$, -2 -2

$x = -2, -2, 2, 2$

$(-2, 0)\ (-2, 0)\ (2, 0)\ (2, 0)$



D: 4 $\quad$ LC: -2 $\quad$ EVEN/NEG.

$f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$; $f(x) \rightarrow -\infty$ as $x \rightarrow +\infty$

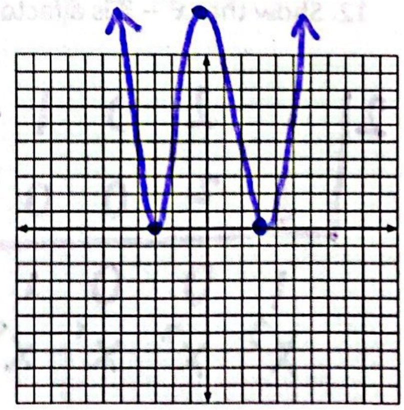
3. Find the zeros of $f(x) = x^4 - 18x^2 + 81$. Then sketch a graph of the function.

$x^4 - 18x^2 + 81 = 0$ $\quad (0, 81)$

$(x^2 - 9)(x^2 - 9) = 0$ $\quad$ D: 4 $\quad$ LC: 1

$(x-3)(x+3)(x-3)(x+3) = 0$ $\quad$ EVEN / POS

$x - 3 = 0 \quad x + 3 = 0 \quad x - 3 = 0 \quad x + 3 = 0$



$x = -3, -3, 3, 3$

$(-3, 0)\ (-3, 0)\ (3, 0)\ (3, 0)$

$f(x) \rightarrow +\infty$ as $x \rightarrow -\infty$; $f(x) \rightarrow +\infty$ as $x \rightarrow +\infty$

Core Concept

The Rational Root Theorem

If $f(x) = a_n x^n + \cdots + a_1 x + a_0$ has *integer* coefficients, then every rational solution of $f(x) = 0$ has the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

- The Rational Root Theorem lists *only possible* solutions
  - To find the ACTUAL solutions, you must test these values via *Synthetic Division*

CONCEPT 3: USING THE RATIONAL ROOT THEOREM

4. Find all real solutions of $x^3 - 8x^2 + 11x + 20 = 0$

q p

$\frac{p}{q} = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{4}{1}, \pm 5, \pm 10, \pm 20$

~~$x=1$~~

1	1	-8	11	20
	↓	1	-7	4
	1	-7	4	R24

$x=-1$

-1	1	-8	11	20
	↓	-1	9	-20
	1	-9	20	R0
	x^2	x^1	x^0	←

$x^2 - 9x + 20 = 0$

$(x-4)(x-5) = 0$

$x-4=0$ $\quad$ $x-5=0$
$+4 \; +4$ $\quad$ $+5 \; +5$

$\boxed{x = -1, 4, 5}$

5. Find all real solutions of $x^3 - 2x^2 - 5x + 6 = 0$

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6$

$x=1$

1	1	-2	-5	6
	↓	1	-1	-6
	1	-1	-6	R0
	x^2	x^1	x^0	←

$x^2 - x - 6 = 0$

$(x+2)(x-3) = 0$

$x+2=0$ $\quad$ $x-3=0$
$-2 \; -2$ $\quad$ $+3 \; +3$

$\boxed{x = 1, -2, 3}$

****CONCEPT 4: USING ZEROS TO WRITE A POLYNOMIAL FUNCTION****

6. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros −2, 3 and 6.

$f(x) = 1(x-(-2))(x-3)(x-6)$

$x^2 - 6x - 3x + 18$

$(x+2)(x^2 - 9x + 18)$

$x^3 - 9x^2 + 18x$
$2x^2 - 18x + 36$

$\boxed{f(x) = x^3 - 7x^2 + 36}$

7. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros −4, 1 and 2.

$f(x) = (x-(-4))(x-1)(x-2)$

$x^2 - 2x - x + 2$

$(x+4)(x^2 - 3x + 2)$

$x^3 - 3x^2 + 2x$
$+4x^2 - 12x + 8$

$\boxed{f(x) = x^3 + x^2 - 10x + 8}$

Core Concept

The Irrational Conjugates Theorem

Let f be a polynomial function with rational coefficients, and let a and b be rational numbers such that $\sqrt{b}$ is irrational. If $a + \sqrt{b}$ is a zero of f, then $a - \sqrt{b}$ is also a zero of f.

7. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 3 and $2 + \sqrt{5}$. 3 $2 - \sqrt{5}$

$f(x) = (x-3)[x-(2+\sqrt{5})][x-(2-\sqrt{5})]$

$(x-3)[(x-2) - \sqrt{5}][(x-2) + \sqrt{5}]$

$(x-3)[(x-2)^2 - (\sqrt{5})^2]$

$(x-3)[x^2 - 4x + 4 - 5]$

$(x-3)(x^2 - 4x - 1)$

$x^3 - 4x^2 - x$
$-3x^2 + 12x + 3$

$\boxed{f(x) = x^3 - 7x^2 + 11x + 3}$

8. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 2 and $3 + \sqrt{7}$. 3 $3 - \sqrt{7}$

$f(x) = (x-2)[x-(3+\sqrt{7})][x-(3-\sqrt{7})]$

$(x-2)[(x-3) - \sqrt{7}][(x-3) + \sqrt{7}]$

$(x-2)[(x-3)^2 - (\sqrt{7})^2]$

$(x-2)[x^2 - 6x + 9 - 7]$

$(x-2)(x^2 - 6x + 2)$

$x^3 - 6x^2 + 2x$
$-2x^2 + 12x - 4$

$\boxed{f(x) = x^3 - 8x^2 + 14x - 4}$

4.6: The Fundamental Theorem of Algebra

Core Concept

The Fundamental Theorem of Algebra

Theorem If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has at least one solution in the set of complex numbers.

Corollary If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has exactly n solutions provided each solution repeated twice is counted as two solutions, each solution repeated three times is counted as three solutions, and so on.

The table shows several polynomial equations and their solutions, including repeated solutions. Notice that for the last equation, the repeated solution $x = -1$ is counted twice.

Equation	Degree	Solution(s)	Number of solutions
$2x - 1 = 0$	1	$\frac{1}{2}$	1
$x^2 - 2 = 0$	2	$\pm\sqrt{2}$	2
$x^3 - 8 = 0$	3	$2, -1 \pm i\sqrt{3}$	3
$x^3 + x^2 - x - 1 = 0$	3	$-1, -1, 1$	3

In the table, note the relationship between the degree of the polynomial $f(x)$ and the number of solutions of $f(x) = 0$. This relationship is generalized by the *Fundamental Theorem of Algebra*, first proven by German mathematician Carl Friedrich Gauss (1777–1855).

CONCEPT 1: FINDING THE NUMBER OF SOLUTIONS OR ZEROS

1. a) $f(x) = x^3 + 3x^2 + 16x + 48 = 0$ b) $f(x) = x^4 + 6x^3 + 12x^2 + 8x = 0$

3

4

CONCEPT 2: FINDING THE ZEROS OF A FUNCTION

2. Find all zeros of $f(x) = x^5 + x^3 - 2x^2 - 12x - 8$

$\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 8$

$x=1$

$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & 1 & -2 & -12 & -8 \\ & & 1 & 1 & 2 & 0 & -12 \\ \hline & 1 & 1 & 2 & 0 & -12 & R\,{-20} \end{array}$$

$\boxed{x=-1}$

$$\begin{array}{r|rrrrrr} -1 & 1 & 0 & 1 & -2 & -12 & -8 \\ & \downarrow & -1 & 1 & -2 & 4 & 8 \\ \hline & 1 & -1 & 2 & -4 & -8 & R0 \end{array}$$

$\boxed{x=2}$

$$\begin{array}{r|rrrrr} 2 & 1 & -1 & 2 & -4 & -8 \\ & \downarrow & 2 & 2 & 8 & 8 \\ \hline & 1 & 1 & 4 & 4 & R0 \end{array}$$

$x^3 \quad x^2 \quad x^1 \quad x^0$

$x^3 + x^2 + 4x + 4 = 0$

$x^2(x+1) + 4(x+1) = 0$

$(x+1)(x^2+4) = 0$

$x+1=0$

$-1 \quad -1$

$\boxed{x=-1}$

$x^2+4=0$

$-4 \quad -4$

$\sqrt{x^2} = \sqrt{-4}$

$\boxed{x = \pm 2i}$

$\boxed{x = -1, -1, 2, \pm 2i}$

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$

3. Find all zeroes of $f(x) = x^5 + 3x^4 + 9x^3 + 23x^2 - 36$

$\boxed{x=1}$

$1 \mid 1 \quad 3 \quad 9 \quad 23 \quad 0 \quad -36$
$\quad\;\; \downarrow \quad 1 \quad 4 \quad 13 \quad 36 \quad 36$
$\quad\;\; 1 \quad 4 \quad 13 \quad 36 \quad 36 \quad R0$

$\boxed{x=-2}$

$-2 \mid 1 \quad 4 \quad 13 \quad 36 \quad 36$
$\quad\;\; \downarrow \quad -2 \quad -4 \quad -18 \quad -36$
$\quad\;\; 1 \quad 2 \quad 9 \quad 18 \quad R0$
$\quad\;\; x^3 \quad x^2 \quad x^1 \quad x^0$

$x = -1$

$-1 \mid 1 \quad 4 \quad 13 \quad 36 \quad 36$
$\quad\quad -1 \quad -3 \quad -10$
$\quad\;\; 1 \quad 3 \quad 10 \quad -26$

$x^3 + 2x^2 + 9x + 18 = 0$

$x^2(x+2) + 9(x+2) = 0$

$(x+2)(x^2+9) = 0$

$x+2=0$ $\quad$ $x^2+9=0$
$-2 \;\; -2$ $\quad$ $-9 \;\; -9$

$\boxed{x=-2}$ $\quad$ $\sqrt{x^2} = \sqrt{-9}$

$\boxed{x = \pm 3i}$

$\boxed{x = 1, -2, -2, 3i, -3i}$

****CONCEPT 3: USING ZEROS TO WRITE A POLYNOMIAL FUNCTION****

4. Write a polynomial function f of least degree that has rational coefficients, a lead coefficient of 1, and zeros 2 and $3+i$. $\quad 3 \quad 3-i$

$f(x) = (x-2)[x-(3+i)][x-(3-i)]$

$(x-2)[(x-3)-i][(x-3)+i]$

$(x-2)[(x-3)^2 - (i)^2]$

$(x-2)(x^2-6x+9-(-1))$

$(x-2)(x^2-6x+10)$

$x^3 - 6x^2 + 10x$
$\quad\; -2x^2 + 12x - 20$

$\boxed{f(x) = x^3 - 8x^2 + 22x - 20}$

5. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and zeros 5 and $1+i$. $\quad 1-i$

$f(x) = (x-5)[x-(1+i)][x-(1-i)]$

$(x-5)[(x-1)-i][(x-1)+i]$

$(x-5)[(x-1)^2 - (i)^2]$

$(x-5)(x^2-2x+1-(-1))$

$(x-5)(x^2-2x+2)$

$x^3 - 2x^2 + 2x$
$\quad\; -5x^2 + 10x - 10$

$\boxed{f(x) = x^2 - 7x^2 + 12x - 10}$

4.7: Transformations of Polynomials (pg. 206-208)

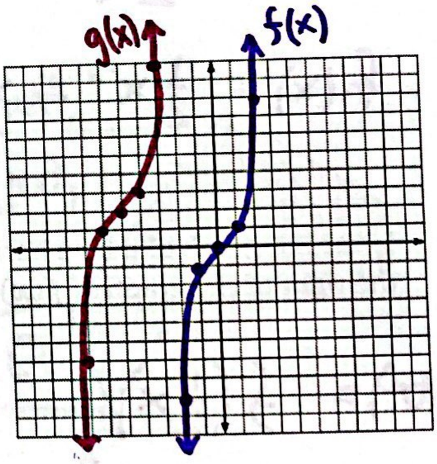
Core Concept

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x-h)$	$g(x)=(x-5)^4$ $g(x)=(x+2)^4$	5 units right 2 units left
Vertical Translation Graph shifts up or down.	$f(x)+k$	$g(x)=x^4+1$ $g(x)=x^4-4$	1 unit up 4 units down
Reflection Graph flips over x- or y-axis.	$f(-x)$ $-f(x)$	$g(x)=(-x)^4=x^4$ $g(x)=-x^4$	over y-axis over x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis.	$f(ax)$	$g(x)=(2x)^4$ $g(x)=\left(\frac{1}{2}x\right)^4$	shrink by a factor of $\frac{1}{2}$ stretch by a factor of 2
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis.	$a \cdot f(x)$	$g(x)=8x^4$ $g(x)=\frac{1}{4}x^4$	stretch by a factor of 8 shrink by a factor of $\frac{1}{4}$

CONCEPT 1: TRANSLATING A POLYNOMIAL FUNCTION

1. Describe the transformation of $f(x)=x^3$ represented by the $g(x)=(x+5)^3+2$. Then graph each function.

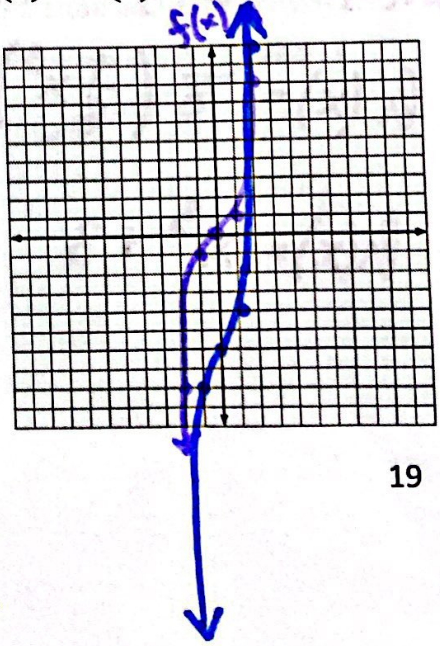
X	-2	-1	0	1	2
f(x)	-8	-1	0	1	8



G(x) IS A HOR. SHIFT 5 UNITS LEFT & A VERT. SHIFT 2 UNITS UP.

2. Describe the transformation of $f(x)=x^3$ represented by the $g(x)=2(x)^3-6$. Then graph each function.

G(x) IS A VERT. STRETCH BY A FACTOR OF 2 AND A VERT. SHIFT 6 UNITS DOWN.



X	-2	-1	0	1	2
G(x)	-22	-8	-6	-4	10

****CONCEPT 2: WRITING A TRANSFORMED POLYNOMIAL FUNCTION****

3. Let the graph of g be a vertical stretch by a factor of 2, followed by a translation 3 units up of the graph of $f(x) = x^4 - 2x^2$. Write a rule for g.

$h(x) = 2 \cdot f(x)$

$h(x)\ 2(x^4 - 2x^2)$

$h(x) = 2x^4 - 4x^2$

$g(x) = h(x) + 3$

$\boxed{g(x) = 2x^4 - 4x^2 + 3}$

4. Let the graph of g be a vertical shrink by a factor of $\frac{1}{2}$, followed by a translation 4 units down of the graph of $f(x) = x^5 - 2x^2$. Write a rule for g.

$h(x) = \frac{1}{2} f(x)$

$h(x) = \frac{1}{2}(x^5 - 2x^2)$

$h(x) = \frac{1}{2}x^5 - x^2$

$g(x) = h(x) - 4$

$\boxed{g(x) = \frac{1}{2}x^5 - x^2 - 4}$

5. Let the graph of g be a horizontal stretch by a factor of 2, followed by a translation 3 units to the right of the graph of $f(x) = 8x^3 + 3$. Write a rule for g.

1
1 1
1 2 1
1 3 3 1

$h(x) = f(\frac{1}{2}x)$

$h(x) = 8(\frac{1}{2}x)^3 + 3$

$h(x) = \cancel{8}(\frac{1}{\cancel{8}}x^3) + 3$

$h(x) = x^3 + 3$

$g(x) = h(x-3)$

$g(x) = (x-3)^3 + 3$

$g(x) = x^3 + 3(x)^2(-3) + 3(x)(-3)^2 + (-3)^3 + 3$

$g(x) = x^3 - 9x^2 + 27x - 27 + 3$

$\boxed{g(x) = x^3 - 9x^2 + 27x - 24}$

6. Let the graph of g be a vertical stretch by a factor of 2, followed by a translation 4 units to the right of the graph of $f(x) = \frac{1}{2}x^3 + x$. Write a rule for g.

$h(x) = 2\,f(x)$

$h(x) = 2(\frac{1}{2}x^3 + x)$

$h(x) = x^3 + 2x$

$g(x) = h(x-4)$

$g(x) = (x-4)^3 + 2(x-4)$

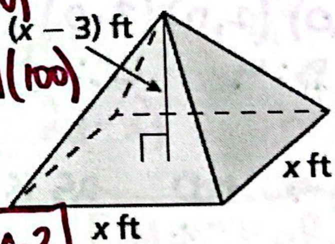
$g(x) = x^3 + 3(x)^2(-4) + 3(x)(-4)^2 + (-4)^3 + 2x - 8$

$g(x) = x^3 - 12x^2 + 48x - 64 + 2x - 8$

$\boxed{g(x) = x^3 - 12x^2 + 50x - 72}$

****CONCEPT 3: MODELING WITH MATHEMATICS****

7. The function $V(x) = \frac{1}{3}x^3 - x^2$ represents the volume (in cubic feet) of the square pyramid shown. The function $W(x) = V(3x)$ represents the volume (in cubic feet) when x is measured in yards. Write a rule for W. Find and interpret $W(10)$.



$W(x) = V(3x)$

$W(x) = \frac{1}{3}(3x)^3 - (3x)^2$

$W(x) = \frac{1}{3}(27x^3) - 9x^2$

$\boxed{W(x) = 9x^3 - 9x^2}$

$W(10) = 9(10)^3 - 9(10)^2$

$W(10) = 9(1000) - 9(100)$

$= 9000 - 900$

$\boxed{W(10) = 8100 \text{ ft}^3}$

8. The function $V(x) = x^3$ represents the volume (in cubic feet) of a cube with side length x. The function $W(x) = V\left(\frac{1}{12}x\right)$ represents the volume (in cubic feet) when x is measured in inches. Write a rule for W. Find and interpret $W(96)$.

$W(x) = V(\frac{1}{12}x)$

$W(x) = (\frac{1}{12}x)^3$

$W(x) = \frac{1}{1728}x^3$

$W(96) = \frac{1}{1728}(96)^3$

$W(96) = \frac{1}{1728}(884{,}736)$

$\boxed{W(96) = 512 \text{ ft}^3}$

4.8: Analyzing Graphs of Polynomials (pg. 212-215)

Concept Summary

Zeros, Factors, Solutions, and Intercepts

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial function. The following statements are equivalent.

Zero: k is a zero of the polynomial function f.

Factor: $x - k$ is a factor of the polynomial $f(x)$.

Solution: k is a solution (or root) of the polynomial equation $f(x) = 0$.

x-Intercept: If k is a real number, then k is an x-intercept of the graph of the polynomial function f. The graph of f passes through $(k, 0)$.

CONCEPT 1: USING X-INTERCEPTS TO GRAPH A POLYNOMIAL

1. Graph the function $f(x) = \frac{1}{6}(x+3)(x-2)^3$

$\frac{1}{6}(x+3)(x-2)^3 = 0$

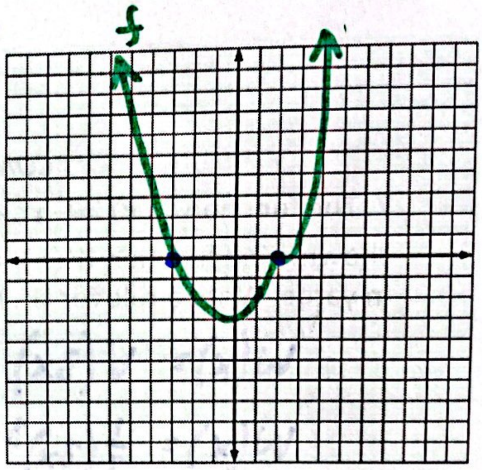
$\frac{1}{6} \neq 0 \quad x+3=0 \quad x-2=0 \quad x-2=0 \quad x-2=0$

$-3 \; -3 \quad +2 \; +2 \quad +2 \; +2 \quad +2 \; +2$

$x = -3, 2, 2, 2$

$(-3,0)\ (2,0)\ (2,0)\ (2,0)$

D: 4 LC: $\frac{1}{6}$ EVEN/Pos.



$f(x) \to +\infty$ as $x \to -\infty$; $f(x) \to +\infty$ as $x \to +\infty$

2. Graph the function $f(x) = \frac{1}{2}(x+1)(x-4)^2$

$\frac{1}{2}(x+1)(x-4)^2 = 0$

$\cancel{\frac{1}{2} = 0} \quad x+1=0 \quad \sqrt{(x-4)^2} = \sqrt{0}$

$-1 \; -1 \quad x-4 = \pm 0$

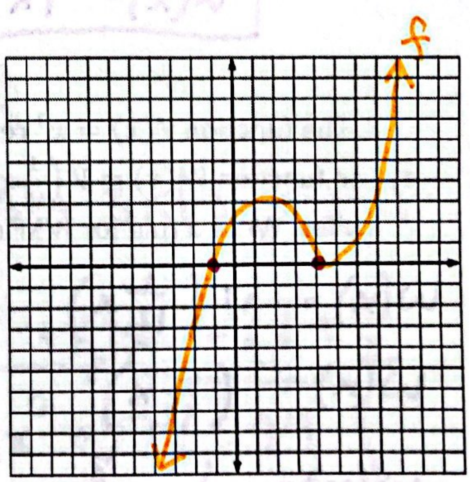
$x = -1 \quad +4 \; +4$

$(-1,0) \quad x = 4, 4$

$(4,0)(4,0)$

D: 3 LC: $\frac{1}{2}$

ODD / Pos.



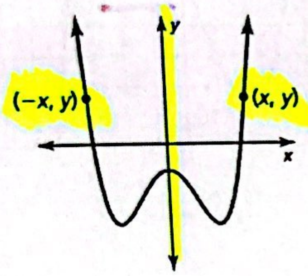
$f(x) \to -\infty$ as $x \to -\infty$; $f(x) \to +\infty$ as $x \to +\infty$

Even and Odd Functions

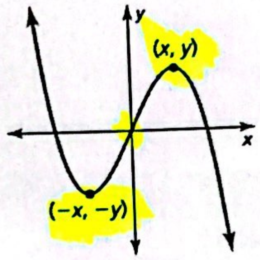
A function f is an **even function when** $f(-x) = f(x)$ for all x in its domain. The graph of an even function is *symmetric about the y-axis.*

A function f is an **odd function when** $f(-x) = -f(x)$ for all x in its domain. The graph of an odd function is *symmetric about the origin.* One way to recognize a graph that is symmetric about the origin is that it looks the same after a 180° rotation about the origin.

Even Function



Odd Function



For an even function, if (x, y) is on the graph, then $(-x, y)$ is also on the graph.

For an odd function, if (x, y) is on the graph, then $(-x, -y)$ is also on the graph.

CONCEPT 2: IDENTIFYING EVEN AND ODD FUNCTIONS

3. Determine whether each function is *even, odd, or neither.*

a) $f(x) = x^3 - 7x$

$f(-x) = (-x)^3 - 7(-x)$

$f(-x) = -x^3 + 7x$

$f(-x) = -(x^3 - 7x)$

$f(-x) = -f(x)$

$\boxed{\text{ODD FUNCTION}}$

b) $g(x) = x^4 + x^2 - 1$

$g(-x) = (-x)^4 + (-x)^2 - 1$

$g(-x) = x^4 + x^2 - 1$

$g(-x) = g(x)$

$\boxed{\text{EVEN FUNCTION}}$

c) $h(x) = x^3 + 2$

$h(-x) = (-x)^3 + 2$

$h(-x) = -x^3 + 2$

$\boxed{\text{NEITHER}}$

4. Determine whether the function is *even, odd, or neither.*

a) $f(x) = -x^2 + 5$

$f(-x) = -(-x)^2 + 5$

$f(-x) = -x^2 + 5$

$f(-x) = f(x)$

$\boxed{\text{EVEN}}$

b) $g(x) = x^4 - 5x^3 - 7$

$g(-x) = (-x)^4 - 5(-x)^3 - 7$

$g(-x) = x^4 + 5x^3 - 7$

$\boxed{\text{NEITHER}}$

c) $h(x) = 2x^5 - 12x$

$h(-x) = 2(-x)^5 - 12(-x)$

$h(-x) = -2x^5 + 12x$

$h(-x) = -(2x^5 - 12x)$

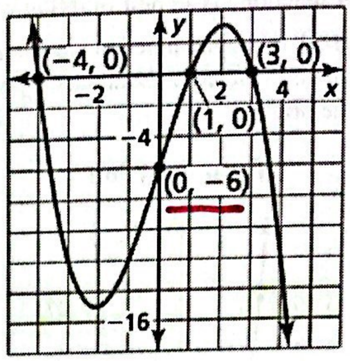
$h(-x) = -h(x)$

$\boxed{\text{ODD}}$

4.9: Modeling with Polynomial Functions (pg. 220-222)

****CONCEPT 1: WRITING A CUBIC FUNCTION*****

1. Write the cubic function whose graph is shown.



$y = a(x-k)(x-k)(x-k)$ $\quad \overset{x,y}{(0,-6)}$ $\quad k = -4, 1, 3$

$-6 = a(0+4)(0-1)(0-3)$

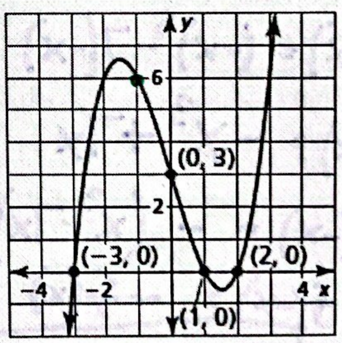
$-6 = a(4)(-1)(-3)$

$\frac{12a}{12} = \frac{-6}{12}$

$a = -\frac{1}{2} = -0.5$

$$\boxed{y = -\tfrac{1}{2}(x+4)(x-1)(x-3) \quad \text{OR} \quad f(x) = -0.5(x+4)(x-1)(x-3)}$$

2. Write the cubic function whose graph is shown.



$y = a(x-k)(x-k)(x-k)$ $\quad \overset{x,y}{(0,3)}$ $\quad k = -3, 1, 2$

$3 = a(0+3)(0-1)(0-2)$

$3 = a(3)(-1)(-2)$

$\frac{6a}{6} = \frac{3}{6}$

$a = \frac{1}{2}$

$$\boxed{f(x) = \tfrac{1}{2}(x+3)(x-1)(x-2)}$$