Chapter 4 Polynomial Functions

4.1: Graphing Polynomial Functions (pg. 158-161)

- A polynomial is an expression of algebraic terms
  - Polynomials often consist of the sum of several terms that contain different powers of the same variable(s)
- A Polynomial Function is a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
  - a_n is the leading coefficient
  - n is the degree
  - a_0 is the constant term
  - Exponents must be whole numbers (i.e. 0, 1, 2, 3, 4, 5...)
  - Coefficients must be real numbers (no imaginary terms)

Common Polynomial Functions

Degree	Type	Standard Form	Example
0	Constant	$f(x) = a_0$	$f(x) = -14$
1	Linear	$f(x) = a_1 x + a_0$	$f(x) = 5x - 7$
2	Quadratic	$f(x) = a_2x^2 + a_1x + a_0$	$f(x) = 2x^2 + x - 9$
3	Cubic	$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$	$f(x) = x^3 - x^2 + 3x$
4	Quartic	$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$	$f(x) = x^4 + 2x - 1$

*****CONCEPT 1: IDENTIFYING POLYNOMIAL FUNCTIONS*****

1. Decide whether each function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

a) $f(x) = 2x^3 + 5x + 8$ ← S.F

D: 3 T: CUBIC L.C.: 2

b) $g(x) = -0.8x^3 + \sqrt{2}x^4 - 12$

S.F. $g(x) = \sqrt{2}x^4 - 0.8x^3 - 12$

D: 4 T: QUARTIC LC: $\sqrt{2}$

c) $h(x) = -x^2 + 7x^{-1} + 4x$

NOT A POLYNOMIAL

d) $k(x) = x^2 + 3^x$

NOT A POLYNOMIAL

Decide whether each function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

2. $f(x) = 7 - 1.6x^2 - 5x$

SF: $f(x) = -1.6x^2 - 5x + 7$

D: 2

T: QUADRATIC

LC: -1.6

3. $p(x) = x + 2x^{-2} + 9.5$

NOT A POLYNOMIAL

4. $q(x) = x^3 - 6x + 3x^4$

SF: $q(x) = 3x^4 + x^3 - 6x$

D: 4

T: QUARTIC

LC: 3

CONCEPT 2: EVALUATING A POLYNOMIAL FUNCTION

5. Evaluate $f(x) = 2x^4 - 8x^2 + 5x - 7$ when $x = 3$.

$f(3) = 2(3)^4 - 8(3)^2 + 5(3) - 7$

$= 162 - 72 + 15 - 7$

$f(3) = 98$

6. Evaluate $f(x) = -2x^4 + 6x^3 - 3x + 11$ when $x = 4$.

$f(4) = -2(4)^4 + 6(4)^3 - 3(4) + 11$

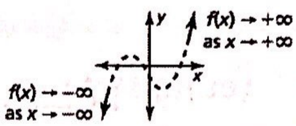
$= -512 + 384 - 12 + 11$

$f(4) = -129$

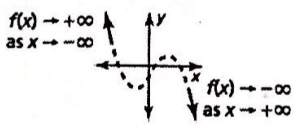
- End Behavior: The behavior of a function's graph as *x* approaches positive infinity $(+\infty)$ or negative infinity $(-\infty)$
  - For polynomial functions, the *end behavior is determined by the* ***function's degree*** *and the* ***sign of the leading coefficient***

End Behavior of Polynomial Functions

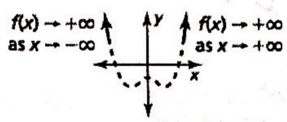
Degree: odd
Leading coefficient: positive



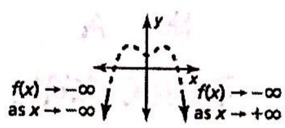
Degree: odd
Leading coefficient: negative



Degree: even
Leading coefficient: positive



Degree: even
Leading coefficient: negative



CONCEPT 3: DESCRIBING END BEHAVIOR

7. Describe the end behavior of the graph Evaluate $f(x) = -0.5x^4 + 2.5x^2 + x - 1$

D: 4 LC: -0.5

EVEN / NEGATIVE

$f(x) \to -\infty$ as $x \to -\infty$; $f(x) \to -\infty$ as $x \to +\infty$

↓ ← ↓ →

8. Describe the end behavior of the graph Evaluate $f(x) = -0.3x^3 + 1.7x^2 - 4x + 6$

D: 3 LC: -0.3

* ODD / NEG

$f(x) \to +\infty$ as $x \to -\infty$; $f(x) \to -\infty$ as $x \to +\infty$

↑ ← ↓ →

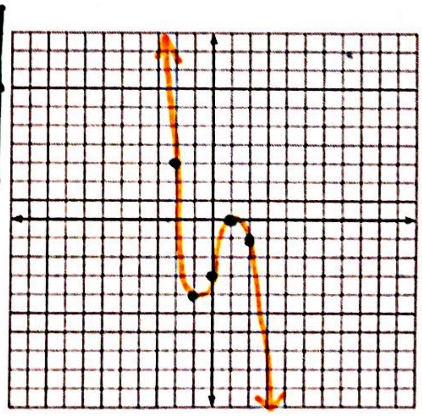
Graphing Polynomial Functions

To graph a polynomial function, first plot points to determine the shape of the graph's middle portion. Then connect the points with a smooth continuous curve and use what you know about end behavior to sketch the graph.

CONCEPT 4: GRAPHING POLYNOMIAL FUNCTIONS

10. Graph (a) $f(x) = -x^3 + x^2 + 3x - 3$

x	-2	-1	0	1	2
f(x)	3	-4	-3	0	-1



$f(-2) = -(-2)^3 + (-2)^2 + 3(-2) - 3 = 8 + 4 - 6 - 3 = 3$

$f(-1) = -(-1)^3 + (-1)^2 + 3(-1) - 3 = 1 + 1 - 3 - 3 = -4$

$f(1) = -(1)^3 + (1)^2 + 3(1) - 3 = -1 + 1 + 3 - 3 = 0$

$f(2) = -(2)^3 + (2)^2 + 3(2) - 3 = -8 + 4 + 6 - 3 = -1$

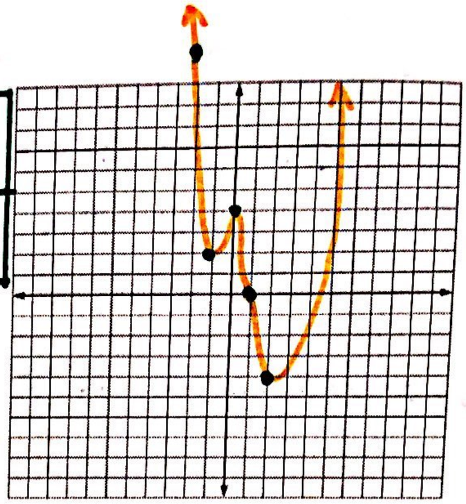
D: 3 LC: -1 ODD/NEG

$f(x) \to +\infty$ as $x \to -\infty$; $f(x) \to -\infty$ as $x \to +\infty$

↑ ← ↓ →

(b) $f(x) = x^4 - x^3 - 4x^2 + 4$

x	-2	-1	0	1	2
f(x)	12	2	4	0	-4

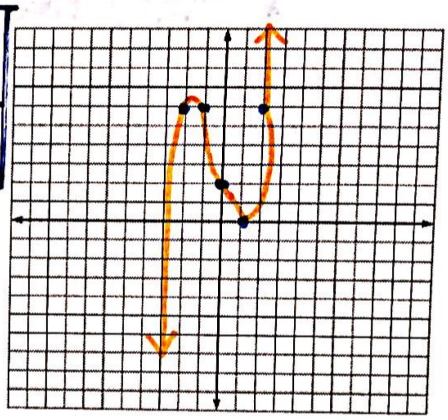


D: 4 LC: 1 EVEN / POS.

$f(x) \to +\infty$ as $x \to -\infty$; $f(x) \to +\infty$ as $x \to +\infty$

11. Graph $f(x) = x^3 + x^2 - 4x + 2$

x	-2	-1	0	1	2
f(x)	6	6	2	0	6



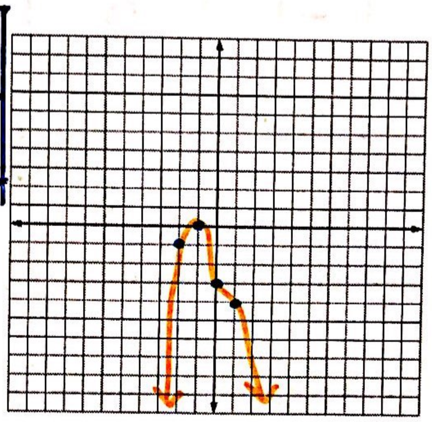
D: 3 LC: 1 ODD / POS.

$f(x) \to -\infty$ as $x \to -\infty$;

$f(x) \to +\infty$ as $x \to +\infty$

12. Graph $f(x) = -x^4 - x^3 + 2x^2 - x - 3$

x	-2	-1	0	1	2
f(x)	-1	0	-3	-4	-21



D: 4 LC: -1 EVEN / NEG.

$f(x) \to -\infty$ as $x \to -\infty$;

$f(x) \to -\infty$ as $x \to +\infty$

4.2 Adding, Subtracting, and Multiplying Polynomials (pg. 166-169)

****CONCEPT 1: ADDING POLYNOMIALS (pg. 166)****

1. (a) Add $3x^3 + 2x^2 - x - 7$ and $x^3 - 10x^2 + 8$

$(3x^3 + 2x^2 \boxed{-x} - 7) + (x^3 - 10x^2 + 8)$

$\boxed{4x^3 - 9x^2 - x + 1}$

****CONCEPT 2: SUBTRACTING POLYNOMIALS (pg. 166)****

2. Subtract (a) $(8x^3 - 3x^2 - 2x + 9) + (-2x^3 - 6x^2 - x - 1)$

$\boxed{6x^3 - 9x^2 - x + 8}$

(b) $3z^2 - z - 4$ from $2z^2 + 3z$

$2z^2 + 3z + (-3z^2 + z + 4)$

$\boxed{-z^2 + 4z + 4}$

3. $(2x^2 - 6x + 5) + (7x^2 - x - 9)$

$\boxed{9x^2 - 7x - 4}$

4. $(3x^3 - 8x^2 - x - 4) + (-5x^3 + x^2 - 17)$

$\boxed{-2x^3 - 7x^2 - x - 21}$

****CONCEPT 3: MULTIPLYING POLYNOMIALS****

5. Multiply $y + 5$ and $3y^2 - 2y + 2$

$3y^3 - 2y^2 + 2y$

$+ \quad 15y^2 - 10y + 10$

$\boxed{3y^3 + 13y^2 - 8y + 10}$

6. Multiply $y-2$ and $2y^2+3y+5$

$2y^3+3y^2+5y$
$+ \quad -4y^2-6y-10$

$\boxed{2y^3-y^2-y-10}$

CONCEPT 4: MULTIPLYING THREE BINOMIALS

7. Multiply $x-1$, $x+4$, and $x+5$

$(x-1)(x+4)$

$x^2+4x-x-4$

x^2+3x-4

$(x+5)(x^2+3x-4)$

x^3+3x^2-4x
$+ \quad 5x^2+15x-20$

$\boxed{x^3+8x^2+11x-20}$

8. Multiply $(x+2)(x-1)$ and $(x-3)$

$x^2-x+2x-2$

$(x-3)(x^2+x-2)$

x^3+x^2-2x
$-3x^2-3x+6$

$\boxed{x^3-2x^2-5x+6}$

Special Product Patterns

Sum and Difference	Example
$(a+b)(a-b) = a^2 - b^2$	$(x+3)(x-3) = x^2 - 9$

Square of a Binomial	Example
$(a+b)^2 = a^2 + 2ab + b^2$	$(y+4)^2 = y^2 + 8y + 16$
$(a-b)^2 = a^2 - 2ab + b^2$	$(2t-5)^2 = 4t^2 - 20t + 25$

Cube of a Binomial	Example
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	$(z+3)^3 = z^3 + 9z^2 + 27z + 27$
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$	$(m-2)^3 = m^3 - 6m^2 + 12m - 8$

****CONCEPT 6: USING SPECIAL PRODUCT PATTERNS****

9. Find the product.

a) $(4n + 5)(4n - 5)$

$16n^2 - 25$

b) $(9y - 2)^2$

$(9y)^2 + 2(9y)(-2) + (-2)^2$

$81y^2 - 36y + 4$

c) $(ab + 4)^3$

$(ab)^3 + 3(ab)^2(4) + 3(ab)(4^2) + 4^3$

$a^3b^3 + 12a^2b^2 + 48ab + 64$

10. a) $(2n + 7)(2n - 7)$

$4n^2 - 49$

b) $(4y + 3)^2$

$16y^2 + 24y + 9$

c) $(ab + 2)^3$

$(ab)^3 + 3(ab)^2(2) + 3(ab)(2)^2 + 2$

$a^3b^3 + 6a^2b^2 + 12ab + 8$

Pascal's Triangle

In Pascal's Triangle, the first and last numbers in each row are 1. Every number other than 1 is the sum of the closest two numbers in the row directly above it. The numbers in Pascal's Triangle are the same numbers that are the coefficients of binomial expansions, as shown in the first six rows.

	n	$(a+b)^n$	Binomial Expansion	Pascal's Triangle
0th row	0	$(a+b)^0 =$	1	1
1st row	1	$(a+b)^1 =$	$1a + 1b$	1 1
2nd row	2	$(a+b)^2 =$	$1a^2 + 2ab + 1b^2$	1 2 1
3rd row	3	$(a+b)^3 =$	$1a^3 + 3a^2b + 3ab^2 + 1b^3$	1 3 3 1
4th row	4	$(a+b)^4 =$	$1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$	1 4 6 4 1
5th row	5	$(a+b)^5 =$	$1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$	1 5 10 10 5 1

****CONCEPT 7: USING PASCAL'S TRIANGLE TO EXPAND BINOMIALS****

11. Use Pascal's Triangle to expand (a) $(x-2)^5$ and (b) $(3y+1)^3$

a) $(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$

$a = x$
$b = -2$

$(x-2)^5 = x^5 + 5(x^4)(-2) + 10(x^3)(-2)^2 + 10(x^2)(-2^3) + 5(x)(-2)^4 + (-2)^5$

$\boxed{(x-2)^5 = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32}$

b) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$a = 3y$
$b = 1$

$(3y+1)^3 = (3y)^3 + 3(3y)^2(1) + 3(3y)(1^2) + 1^3$

$\boxed{(3y+1)^3 = 27y^3 + 27y^2 + 9y + 1}$

12. Use Pascal's Triangle to expand $(x-3)^4$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$a = x$
$b = -3$

$(x-3)^4 = x^4 + 4(x^3)(-3) + 6(x^2)(-3^2) + 4x(-3^3) + (-3^4)$

$\boxed{(x-3)^4 = x^4 - 12x^3 + 54x^2 - 108x + 81}$

13. Use Pascal's Triangle to expand $(2x+4)^3$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$a = 2x$
$b = 4$

$(2x+4)^3 = (2x)^3 + 3(2x)^2(4) + 3(2x)(4^2) + 4^3$

$\boxed{(2x+4)^3 = 8x^3 + 48x^2 + 96x + 64}$

4.3: Dividing Polynomials (pg. 174-176)

<u>Dividing Polynomials</u>

- There are two methods for dividing polynomials. You will <u>need to know both</u>
    1. <u>Long Division</u>
    2. <u>Synthetic Division</u>

1. LONG DIVISON:

When you divide a polynomial $f(x)$ by a nonzero polynomial divisor $d(x)$, you get a quotient polynomial $q(x)$ and a remainder polynomial $r(x)$.

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

CONCEPT 1: USING POLYNOMIAL LONG DIVISION

1. Divide $2x^4 + 3x^3 + 5x - 1$ by $x^2 + 3x + 2$

$2x^2 - 3x + 5$

$x^2+3x+2 \;\overline{)\; 2x^4 + 3x^3 + 5x - 1}$

$-2x^4 + 6x^3 + 4x^2$

$-3x^3 - 4x^2 + 5x$

$+3x^3 + 9x^2 + 6x$

$5x^2 + 11x - 1$

$-5x^2 + 15x + 10$

$-4x - 11$ R

$2x^2 - 3x + 5 + \frac{-4x-11}{x^2+3x+2}$

$\boxed{2x^2 - 3x + 5 - \frac{4x+11}{x^2+3x+2}}$

2. $(x^3 - x^2 - 2x + 8) \div (x - 1)$

$x^2 - 2$

$x-1 \;\overline{)\; x^3 - x^2 - 2x + 8}$

$-x^3 + x^2$

$-2x + 8$

$+2x - 2$

R 6

$\boxed{x^2 - 2 + \frac{6}{x-1}}$

3. $(x^2 + 2x^2 - x + 5) \div (x^2 - x + 1)$

3

$(x^3 + 2x^2 - x + 5) \div (x^2 - x + 1)$

$x + 3$

$x^2 - x + 1 \;\overline{)\; x^3 + 2x^2 - x + 5}$

$-x^3 + x^2 + -x$

$3x^2 - 2x + 5$

$-3x^2 + 3x + -3$

R $x + 2$

$\boxed{x + 3 + \frac{x+2}{x^2-x+1}}$

2. SYNTHETIC DIVISION

- A shortcut for dividing polynomials by binomials of the form *(x - k)*

→How to use Synthetic Division:

(0x) ***CONCEPT 2: USING SYNTHETIC DIVISION***

4. Divide $-x^3 + 4x^2 + 9$ by $x - 3$

$$\begin{array}{r|rrrr} 3 & -1 & 4 & 0 & 9 \\ & \downarrow & -3 & 3 & 9 \\ \hline & -1 & 1 & 3 & R\,18 \end{array}$$

$$\boxed{-x^2 + x + 3 + \frac{18}{x-3}}$$

5. Divide $3x^3 - 2x^2 + 2x$ by $x + 1$

6+5

$3x^3 - 2x^2 + 2x + 5$ by $x+1$

$$\begin{array}{r|rrrr} -1 & 3 & -2 & 2 & 5 \\ & \downarrow & -3 & 5 & -7 \\ \hline & 3 & -5 & 7 & R\,-2 \end{array}$$

$$\boxed{3x^2 - 5x + 7 - \frac{2}{x+1}}$$

Divide using synthetic division

6. $(x^3 - x^2 - 2x + 8) \div (x - 1)$

$$\begin{array}{r|rrrr} 1 & 1 & -1 & -2 & 8 \\ & & 1 & 0 & -2 \\ \hline & 1 & 0 & -2 & R\,6 \end{array}$$

$$\boxed{x^2 - 2 + \frac{6}{x-1}}$$

7. $(2x^3 - x - 7) \div (x + 3)$

$$\begin{array}{r|rrrr} -3 & 2 & 0 & -1 & -7 \\ & & -6 & 18 & -51 \\ \hline & 2 & -6 & 17 & R\,-58 \end{array}$$

$$\boxed{2x^2 - 6x + 17 - \frac{58}{x+3}}$$

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

- You can use the Remainder Theorem to tell whether *synthetic division* can be used to evaluate a polynomial function
- Your goal is to *determine the remainder of the function*

*****CONCEPT 4: EVALUATING A POLYNOMIAL*****

8. Use synthetic division to evaluate $f(x) = 5x^3 - x^2 + 13x + 29;\ x = -4$

$$\begin{array}{r|rrrr} -4 & 5 & -1 & 13 & 29 \\ & \downarrow & -20 & 84 & -388 \\ \hline & 5 & -21 & 97 & R\,-359 \end{array}$$

$\boxed{f(-4) = -359}$

Use synthetic division to evaluate the function for the indicated value of x.

9. $f(x) = 4x^2 - 10x - 21; x = 5$

$$\begin{array}{r|rrr} 5 & 4 & -10 & -21 \\ & & 20 & 50 \\ \hline & 4 & 10 & R\,29 \end{array}$$

$\boxed{f(5) = 29}$

10. $f(x) = 5x^4 + 2x^3 - 20x - 6; x = 2$

$$\begin{array}{r|rrrrr} 2 & 5 & 2 & 0 & -20 & -6 \\ & & 10 & 24 & 48 & 56 \\ \hline & 5 & 12 & 24 & 28 & R\,50 \end{array}$$

$\boxed{f(2) = 50}$

4.4: Factoring Polynomials (pg. 180-182)

Factoring Polynomials

- You've previously factored *Quadratic Equations* of degree 2
- You can also factor Polynomials with degree greater than 2

**Recall that a *factor* is a number that is multiplied to get a product

****CONCEPT 1: FINDING A COMMON MONOMIAL FACTOR****

1. Factor each polynomial completely

a) $x^3 - 4x^2 - 5x$

$x(x^2-4x-5)$

$\boxed{x(x-5)(x+1)}$

b) $3x^5 - 48x^3$

$3x^3(x^2-16)$

$\boxed{3x^3(x-4)(x+4)}$

c) $5x^4 + 30x^3 + 45x^2$

$5x^2(x^2+6x+9)$

$\boxed{5x^2(x+3)^2}$

Factor each polynomial completely

2. $x^3 - 7x^2 + 10x$

$x(x^2-7x+10)$

$\boxed{x(x-2)(x-5)}$

3. $3x^7 - 75x^5$

$3x^5(x^2-25)$

$\boxed{3x^5(x-5)(x+5)}$

Special Factoring Patterns

Sum of Two Cubes

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

Example

$$64x^3 + 1 = (4x)^3 + 1^3$$
$$= (4x+1)(16x^2 - 4x + 1)$$

Difference of Two Cubes

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Example

$$27x^3 - 8 = (3x)^3 - 2^3$$
$$= (3x-2)(9x^2 + 6x + 4)$$

CONCEPT 2: FACTORING THE SUM OR DIFFERENCE OF TWO CUBES

4. Factor (a) $\cancel{3}x^3 - 125$ and (b) $16x^5 + 54x^2$ completely

a) $x^3 - 125$

$\boxed{(x-5)(x^2+5x+25)}$

b) $16x^5 + 54x^2$

$2x^2(8x^3+27)$

$\boxed{2x^2(2x+3)(4x^2-6x+9)}$

5. Factor (a) $x^3 - 64$ and (b) $-16x^5 + 250x^2$ completely

a) $x^3 - 64$

$\boxed{(x-4)(x^2+4x+16)}$

b) $-16x^5 + 250x^2$

$-2x^2(8x^3-125)$

$\boxed{-2x^2(2x-5)(4x^2+10x+25)}$

CONCEPT 3: FACTORING BY GROUPING

6. Factor $x^3 + 5x^2 - 4x - 20$ completely.

$x^2(x+5) - 4(x+5)$

$(x+5)(x^2-4)$

$\boxed{(x+5)(x-2)(x+2)}$

7. Factor $x^3 - 2x^2 - 9x + 18$ completely.

$x^2(x-2) - 9(x-2)$

$(x-2)(x^2-9)$

$\boxed{(x-2)(x-3)(x+3)}$

CONCEPT 4: FACTORING POLYNOMIALS IN QUADRATIC FORM

8. Factor (a) $16x^4 - 81$ and (b) $3x^8 + 15x^3 + 18x^2$ completely.

5

a) $16x^4 - 81$

$(4x^2-9)(4x^2+9)$

$\boxed{(2x-3)(2x+3)(4x^2+9)}$

b) $3x^8 + 15x^5 + 18x^2$

$3x^2(x^6+5x^3+6)$

$\boxed{3x^2(x^3+2)(x^3+3)}$

9. Factor (a) $625x^8 - 256$ and (b) $2x^{13} + 10x^9 + 8x^5$ completely.

a) $625x^8 - 256$

$(25x^4 - 16)(25x^4 + 16)$

$\boxed{(5x^2-4)(5x^2+4)(25x^4+16)}$

b) $2x^{13} + 10x^9 + 8x^5$

$2x^5(x^8 + 5x^4 + 4)$

$\boxed{2x^5(x^4+4)(x^4+1)}$

The Factor Theorem

A polynomial $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

CONCEPT 5: DETERMINING WHETHER A LINEAR BINOMIAL IS A FACTOR

10. Determine whether (a) $x - 2$ is a factor of $f(x) = x^2 + 2x - 4$ and then (b) $x + 5$ is a factor of $f(x) = 3x^4 + 15x^3 - x^2 + 25$

a)

2	1	2	-4
	↓	2	8
	1	4	R4

$\cancel{R=0}$ NOPE!

b)

-5	3	15	-1	0	25
	↓	-15	0	5	-25
	3	0	-1	5	$\boxed{R0}$

FACTORS
↓
$f(x) = (x+5)(3x^3 - x + 5)$

YES! X+5 IS A FACTOR

CONCEPT 6: FACTORING A BINOMIAL

11. Show that $x + 3$ is a factor of $f(x) = x^4 + 3x^3 - x - 3$. Then factor completely.

-3	1	3	0	-1	-3
	↓	-3	0	0	3
	1	0	0	-1	R0

$f(x) = (x+3)(x^3 - 1)$

$\boxed{f(x) = (x+3)(x-1)(x^2+x+1)}$

12. Show that $x - 2$ is a factor of $f(x) = x^4 - 2x^3 + x - 2$. Then factor completely.

2	1	-2	0	1	-2
		2	0	0	2
	1	0	0	1	R0

$f(x) = (x-2)(x^3+1)$

$\boxed{f(x) = (x-2)(x+1)(x^2-x+1)}$