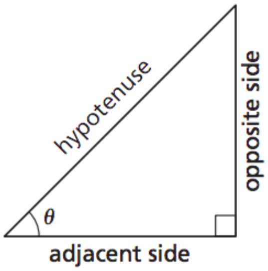
Chapter 9 Trigonometric Ratios and Functions

9.1: Right Angle Trigonometry (pg. 462 – 465)

θ	sinθ	cosθ	tanθ	cscθ	secθ	cotθ
30°						
45°						
60°						



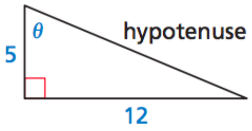
- Every right triangle has an acute angle, θ
- The three sides of a triangle are referenced with respect to θ
- The ratios of these sides are used to define different trigonometric functions:
- The abbreviations ***opp.*** , ***hyp.*** , & ***adj.*** are often used to represent these side lengths

$$\sin\theta = \frac{opposite}{hypotenuse} \qquad \cos\theta = \frac{adjacent}{hypotenuse} \qquad \tan\theta = \frac{opposite}{adjacent}$$

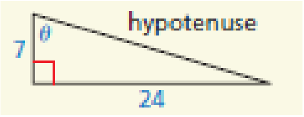
$$\csc\theta = \frac{hypotenuse}{opposite} \qquad \sec\theta = \frac{hypotenuse}{adjacent} \qquad \cot\theta = \frac{adjacent}{opposite}$$

*****CONCEPT 1: EVALUATING TRIG FUNCTIONS GIVEN A TRIANGLE*****

1. Evaluate the six trig functions of the angle θ



2. Evaluate the six trig functions of the angle θ



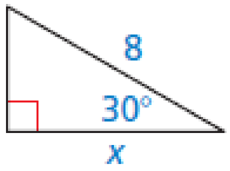
*****CONCEPT 2: EVALUATING TRIG FUNCTIONS GIVEN A TRIG FUNCTION*****

3. In a right triangle, θ is an acute angle and $\mathbf{sin}\ \boldsymbol{\theta} = \frac{\mathbf{4}}{\mathbf{7}}$. Evaluate the other five trigonometric functions of θ.

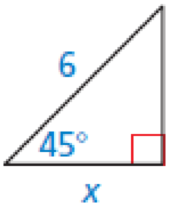
4. In a right triangle, θ is an acute angle and $\mathbf{sin}\ \boldsymbol{\theta} = \frac{\mathbf{5}}{\mathbf{6}}$. Evaluate the other five trigonometric functions of θ.

*****CONCEPT 3: FINDING AN UNKNOWN SIDE LENGTH*****

5. Find the value *x* for the right triangle

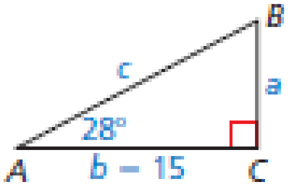


6. Find the value *x* for the right triangle

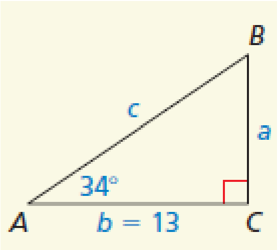


*****CONCEPT 4: USING A CALCULATOR TO SOLVE A RIGHT TRIANGLE*****

7. Solve ΔABC

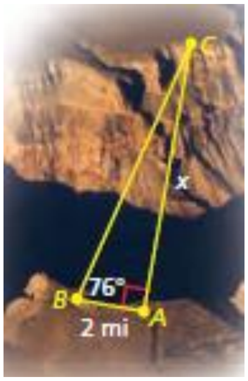


8. Solve ΔABC



*****CONCEPT 5: REAL WORLD PROBLEMS*****

9. You are hiking near a canyon. While standing at A, you measure an angle of 90° between B and C, as shown. You then walk to B and measure an angle of 76° between A and C. The distance between A and B is about 2 miles. How wide is the canyon between A and C?

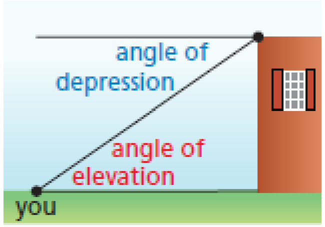


10. You are hiking near a canyon. While standing at A, you measure an angle of 90° between B and C. You then walk to B and measure an angle of 63° between A and C. The distance between A and B is about 5 miles. How wide is the canyon between A and C?

*** CONCEPT 6: USING ANGLE OF ELEVATION***

<u>Angle of Depression:</u> Angle below a horizontal line.

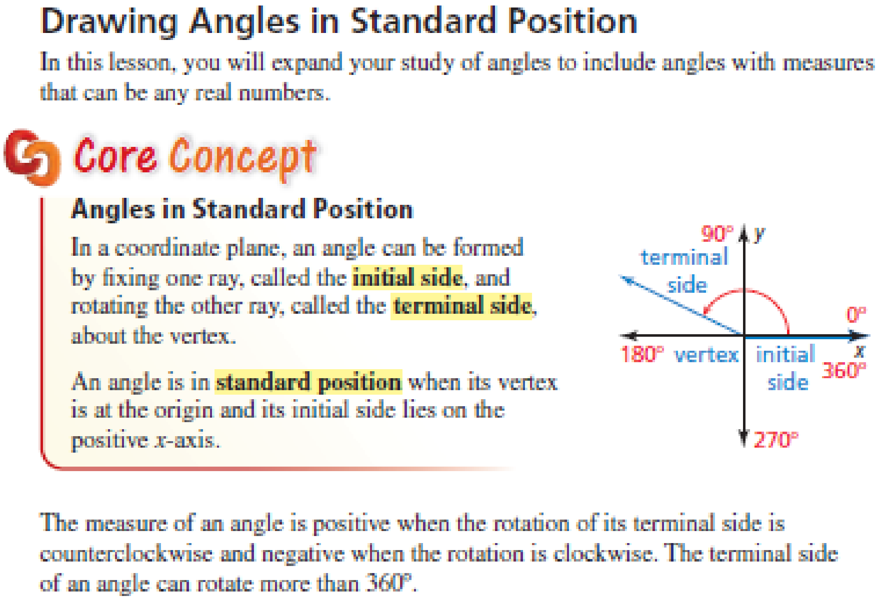
<u>Angle of Elevation:</u> Angle above a horizontal line.



11. A parasailer is attached to a boat with a rope 72 feet long. The angle of elevation from the boat to the parasailer is 28°. Estimate the parasailer's height above the boat.

12. From the top of a lighthouse, an observer notices a boat and finds the angle of depression to be 12°. If the boat is 900 feet from the bottom of the lighthouse, what is the height of the lighthouse?

9.2: Angles and Radian Measures (pg. 470 – 473)



Drawing Angles in Standard Position

In this lesson, you will expand your study of angles to include angles with measures that can be any real numbers.

Core Concept

Angles in Standard Position

In a coordinate plane, an angle can be formed by fixing one ray, called the **initial side**, and rotating the other ray, called the **terminal side**, about the vertex.

An angle is in **standard position** when its vertex is at the origin and its initial side lies on the positive *x*-axis.

The measure of an angle is positive when the rotation of its terminal side is counterclockwise and negative when the rotation is clockwise. The terminal side of an angle can rotate more than 360°.

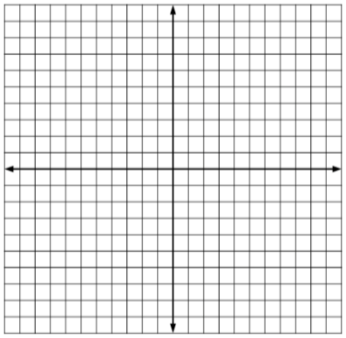
*****CONCEPT 1: DRAWING ANGLES IN STANDARD POSTION*****

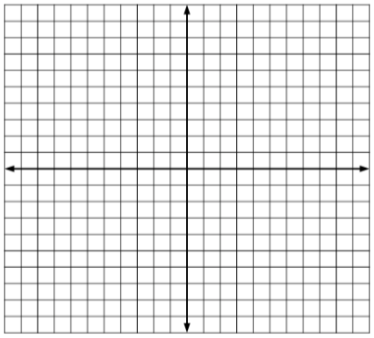
1. Draw an angle with the given measure in standard position.

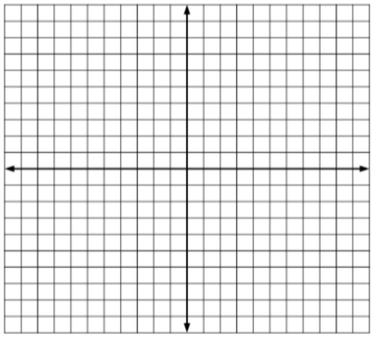
a. 240°

b. 500°

c. −50°





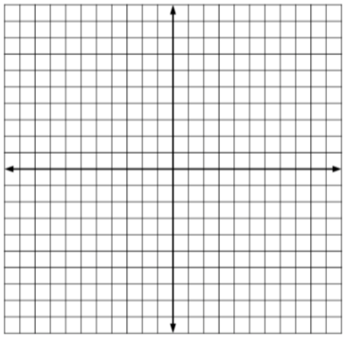


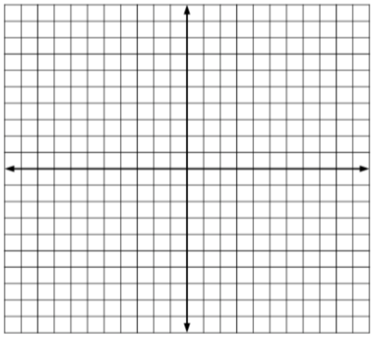
2. Draw an angle with the given measure in standard position.

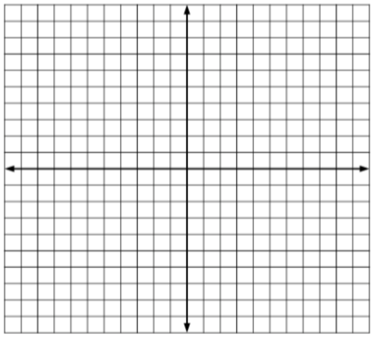
a. 390°

b. -160°

c. 690°

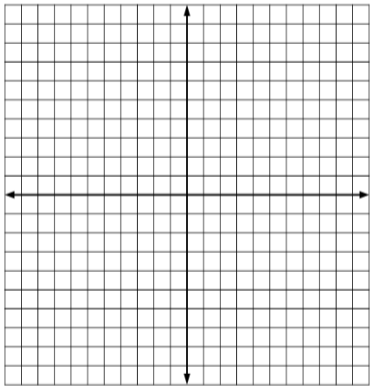


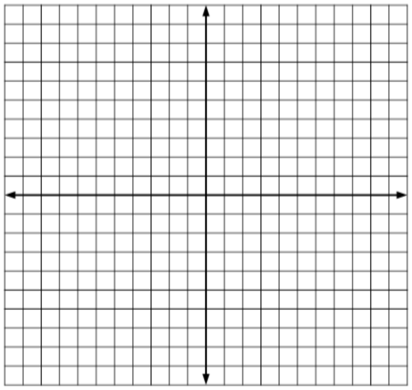




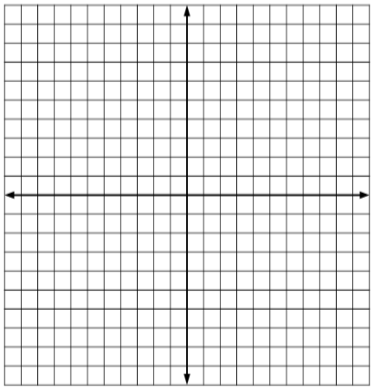
CONCEPT 2: FINDING COTERMINAL ANGLES

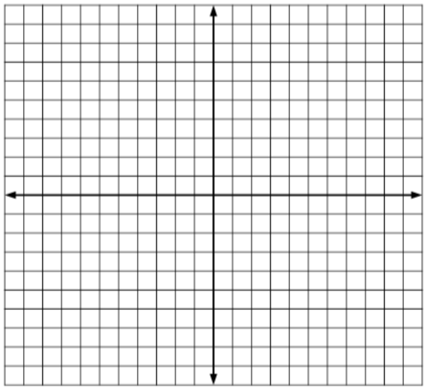
3. Find one positive angle and one negative angle that are coterminal with (a) −45° and (b) 395°.





4. Find one positive angle and one negative angle that are coterminal with (a) −75° and (b) 460°.





Core Concept

Converting Between Degrees and Radians

Degrees to radians

Multiply degree measure by

$$\frac{\pi \text{ radians}}{180^\circ}.$$

Radians to degrees

Multiply radian measure by

$$\frac{180^\circ}{\pi \text{ radians}}.$$

*****CONCEPT 3: COVERT BETWEEN DEGREES AND RADIANS*****

5. Convert the degree measure to radians or the radian measure to degrees.

a. 120°

b. $-\frac{\pi}{12}$

6. Convert the degree measure to radians or the radian measure to degrees.

a. 150°

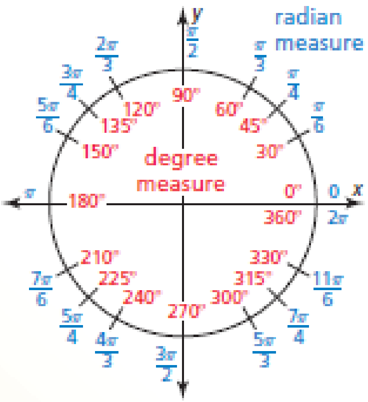
b. $\frac{9\pi}{4}$

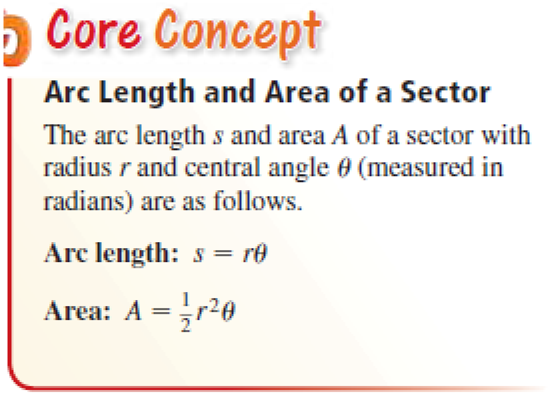
Concept Summary

Degree and Radian Measures of Special Angles

The diagram shows equivalent degree and radian measures for special angles from 0° to 360° (0 radians to 2π radians).

You may find it helpful to memorize the equivalent degree and radian measures of special angles in the first quadrant and for $90^\circ = \frac{\pi}{2}$ radians. All other special angles shown are multiples of these angles.





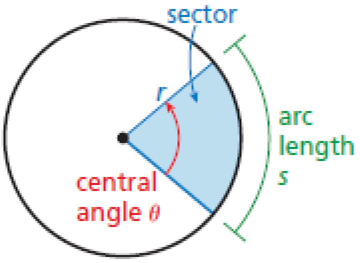
Core Concept

Arc Length and Area of a Sector

The arc length s and area A of a sector with radius r and central angle θ (measured in radians) are as follows.

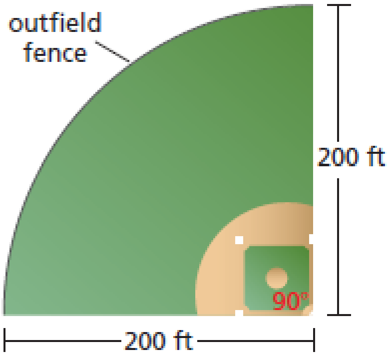
Arc length: $s = r\theta$

Area: $A = \frac{1}{2}r^2\theta$



*****CONCEPT 4: MODELING WITH MATHEMATICS*****

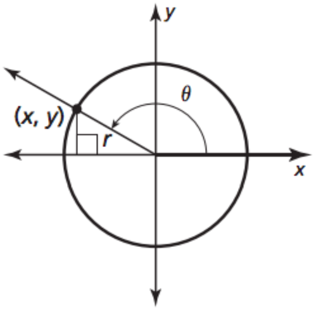
7. A softball field forms a sector with the dimension shown. Find the length of the outfield fence and the area of the field.



8. Like in example 7, suppose the dimensions of the field measure 180 ft, instead of 200 ft. Find the length of the outfield fence and the area of the field.

9.3: Trigonometric Functions of Any Angle (pg. 478 – 481)

Let θ be an angle in standard position, and let (x, y) be the point where the terminal side of θ intersects the circle $x^2 + y^2 = r^2$. The six trigometric functions of θ are defined as shown.



$\sin\theta = \frac{y}{r}$ $\qquad$ $\csc\theta = \frac{r}{y}, y \neq 0$

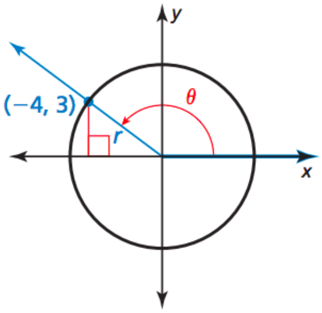
$\cos\theta = \frac{x}{r}$ $\qquad$ $\sec\theta = \frac{r}{x}, x \neq 0$

$\tan\theta = \frac{y}{x}, x \neq 0$ $\qquad$ $\cot\theta = \frac{x}{y}, y \neq 0$

These functions are sometimes called *circular functions.*

*****CONCEPT 1: EVALUATING TRIG FUNCTIONS GIVEN A POINT*****

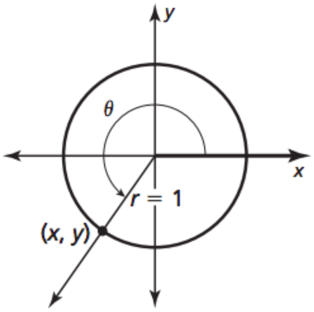
1. Let (-4, 3) be a point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ.



2. Let (8, -6) be a point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ.

The Unit Circle

The circle$x^2 + y^2 = 1$, which has center (0,0) and radius 1, is called the **unit circle.** The values of $\sin\theta$ and $\cos\theta$ are simply the *y*-coordinate and *x*-coordinate, respectively, of the pointy where the terminal side of θ intersects the unit circle.

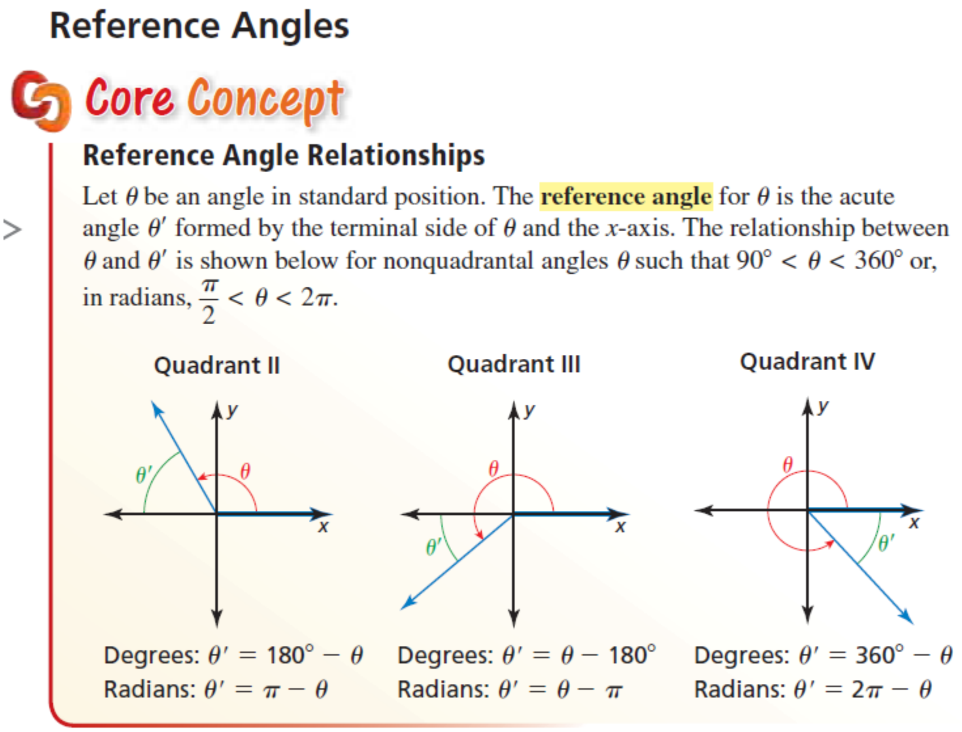


$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y \qquad \cos\theta = \frac{x}{r} = \frac{x}{1} = x$$

*****CONCEPT 2: USING THE UNIT CIRCLE*****

3. Use the unit circle to evaluate the six trig functions of θ*= 270°*

4. Use the unit circle to evaluate the six trigonometric functions of $\theta = 180°$.



Reference Angles

Core Concept

Reference Angle Relationships

Let θ be an angle in standard position. The **reference angle** for θ is the acute angle θ' formed by the terminal side of θ and the x-axis. The relationship between θ and θ' is shown below for nonquadrantal angles θ such that $90° < \theta < 360°$ or, in radians, $\frac{\pi}{2} < \theta < 2\pi$.

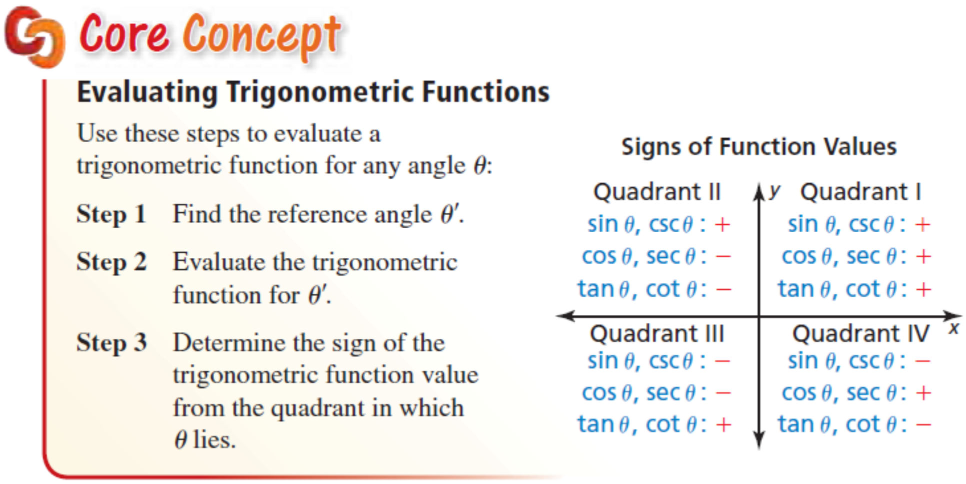
Quadrant II	Quadrant III	Quadrant IV
Degrees: $\theta' = 180° - \theta$	Degrees: $\theta' = \theta - 180°$	Degrees: $\theta' = 360° - \theta$
Radians: $\theta' = \pi - \theta$	Radians: $\theta' = \theta - \pi$	Radians: $\theta' = 2\pi - \theta$

*****CONCEPT 3: FINDING REFERENCE ANGLES*****

Reference Angle: *Angle between terminal side (of θ) and the x-axis (aka horizontal line)*

5. Find the reference angle θ' for (a) $\theta = \frac{5\pi}{3}$ and (b) $\theta = -130°$

6. Find the reference angle θ' for each angle. (a) $\theta = -220°$ and (b) $\frac{3\pi}{4}$



Core Concept

Evaluating Trigonometric Functions

Use these steps to evaluate a trigonometric function for any angle θ:

Step 1 Find the reference angle θ'.

Step 2 Evaluate the trigonometric function for θ'.

Step 3 Determine the sign of the trigonometric function value from the quadrant in which θ lies.

Signs of Function Values

Quadrant II	Quadrant I
$\sin\theta, \csc\theta$: +	$\sin\theta, \csc\theta$: +
$\cos\theta, \sec\theta$: −	$\cos\theta, \sec\theta$: +
$\tan\theta, \cot\theta$: −	$\tan\theta, \cot\theta$: +
Quadrant III	**Quadrant IV**
$\sin\theta, \csc\theta$: −	$\sin\theta, \csc\theta$: −
$\cos\theta, \sec\theta$: −	$\cos\theta, \sec\theta$: +
$\tan\theta, \cot\theta$: +	$\tan\theta, \cot\theta$: −

*****CONCEPT 4: USING REFERENCE ANGLES TO EVALUATE FUNCTIONS*****

7. Evaluate (a) tan (-240°) and (b) csc $\frac{17\pi}{6}$ ***(Could Draw angle, then find reference angle)***

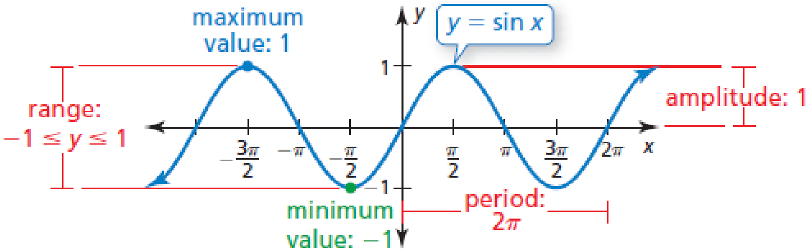
8. Evaluate (a) cot (-210°) and (b) cos $\frac{10\pi}{4}$.

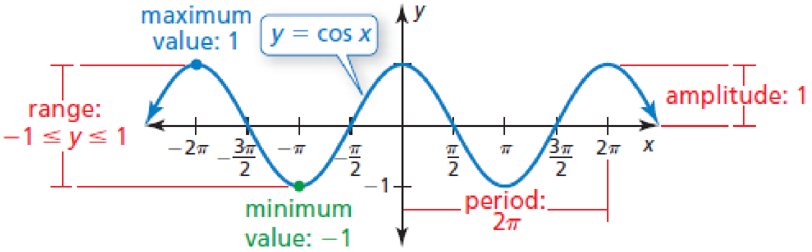
9.4 Graphing Sine & Cosine Functions (pg. 486 – 490)

Exploring Characteristics of Sine and Cosine Functions

In this lesson, you will learn to graph sine and cosine functions. The graphs of sine and cosine functions are related to the graphs of the parent functions $y = \sin x$ and $y = \cos x$, which are shown below.

x	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \sin x$	0	1	0	-1	0	1	0	-1	0
$y = \cos x$	1	0	-1	0	1	0	-1	0	1





Core Concept

Characteristics of $y = \sin x$ and $y = \cos x$

- The domain of each function is all real numbers.
- The range of each function is $-1 \leq y \leq 1$. So, the minimum value of each function is -1 and the maximum value is 1.
- The **amplitude** of the graph of each function is one-half of the difference of the maximum value and the minimum value, or $\frac{1}{2}[1 - (-1)] = 1$.
- Each function is **periodic**, which means that its graph has a repeating pattern. The shortest repeating portion of the graph is called a **cycle**. The horizontal length of each cycle is called the **period**. Each graph shown above has a period of 2π.
- The x-intercepts for $y = \sin x$ occur when $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.
- The x-intercepts for $y = \cos x$ occur when $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$.

Stretching and Shrinking Sine and Cosine Functions

The graphs of $y = a \sin bx$ and $y = a \cos bx$ represent transformations of their parent functions. The value of a indicates a vertical stretch ($a > 1$) or a vertical shrink ($0 < a < 1$) and changes the amplitude of the graph. The value of b indicates a horizontal stretch ($0 < b < 1$) or a horizontal shrink ($b > 1$) and changes the period of the graph.

$$y = a \sin bx$$

$$y = a \cos bx$$

vertical stretch or shrink by a factor of a ↑ ↑ horizontal stretch or shrink by a factor of $\frac{1}{b}$

Core Concept

Amplitude and Period

The amplitude and period of the graphs of $y = a \sin bx$ and $y = a \cos bx$, where a and b are nonzero real numbers, are as follows:

$$\text{Amplitude} = |a| \qquad\qquad \text{Period} = \frac{2\pi}{|b|}$$

CONCEPT 1: GRAPHING A SINE FUNCTION

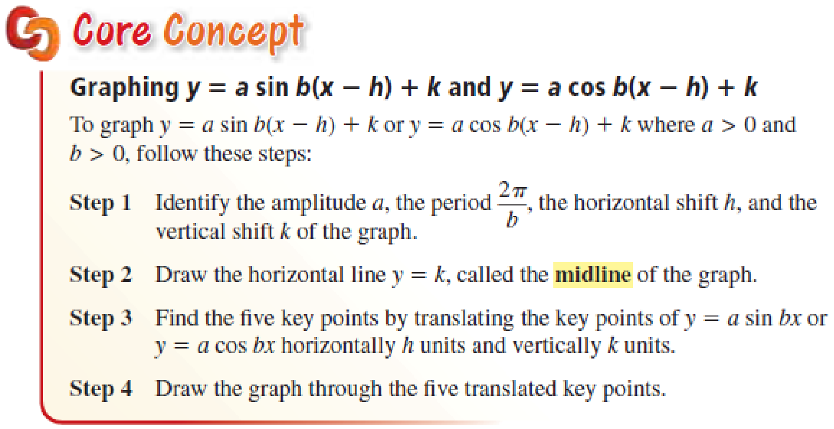
1. Identify the amplitude and period of $g(x) = 4 \sin x$. Then graph the function and describe the graph of g as a transformation of the graph of $f(x) = \sin x$.

2. Identify the amplitude and period of *g*(*x*) = 5 sin *x*. Then graph the function and describe the graph of *g* as a transformation of the graph of *f* (*x*) = sin *x*.

CONCEPT 2: GRAPHING A COSINE FUNCTION

3. Identify the amplitude and period of $g(x) = \frac{1}{2}\cos 2\pi\, x$. Then graph the function and describe the graph of *g* as a transformation of the graph of $f(x) = \cos x$.

4. Identify the amplitude and period of $g(x) = 3\cos \pi x$. Then graph the function and describe the graph of g as a transformation of the graph of $f(x) = \cos x$.



Core Concept

Graphing $y = a \sin b(x - h) + k$ and $y = a \cos b(x - h) + k$

To graph $y = a \sin b(x - h) + k$ or $y = a \cos b(x - h) + k$ where $a > 0$ and $b > 0$, follow these steps:

Step 1 Identify the amplitude a, the period $\frac{2\pi}{b}$, the horizontal shift h, and the vertical shift k of the graph.

Step 2 Draw the horizontal line $y = k$, called the **midline** of the graph.

Step 3 Find the five key points by translating the key points of $y = a \sin bx$ or $y = a \cos bx$ horizontally h units and vertically k units.

Step 4 Draw the graph through the five translated key points.

*****CONCEPT 3: GRAPHING A VERTICAL TRANSLATION*****

5. Graph $g(x) = 2\sin 4x + 3$

6. Graph $g(x) = 3\sin 2x + 5$

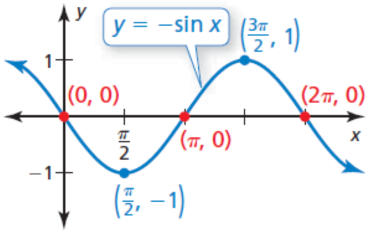
CONCEPT 4: GRAPHING A HORIZONTAL TRANSLATION

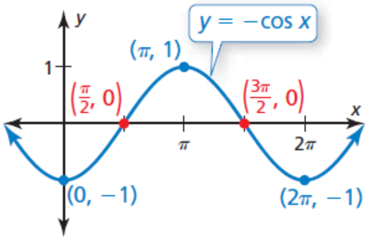
7. Graph $g(x) = 5\cos\frac{1}{2}(x - 3\pi)$

8. Graph $g(x) = 2\cos\frac{1}{4}(x - \pi)$

Reflecting Sine and Cosine Functions

You have graphed functions of the form $y = a \sin b(x - h) + k$ and $y = a \cos b(x - h) + k$, where $a > 0$ and $b > 0$. To see what happens when $a < 0$, consider the graphs of $y = -\sin x$ and $y = -\cos x$.





The graphs are reflections of the graphs of $y = \sin x$ and $y = \cos x$ in the x-axis. In general, when $a < 0$, the graphs of $y = a \sin b(x - h) + k$ and $y = a \cos b(x - h) + k$ are reflections of the graphs of $y = |a| \sin b(x - h) + k$ and $y = |a| \cos b(x - h) + k$, respectively, in the midline $y = k$.

CONCEPT 5: GRAPHING A REFLECTION

9. Graph $g(x) = -2\sin\frac{2}{3}\left(x - \frac{\pi}{2}\right)$

10. Graph $g(x) = -3\sin\frac{1}{2}(x - \pi)$

9.5 Graphing Other Trigonometric Functions (pg. 498 – 501)

Exploring Tangent and Cotangent Functions

The graphs of tangent and cotangent functions are related to the graphs of the parent functions $y = \tan x$ and $y = \cot x$, which are graphed below.

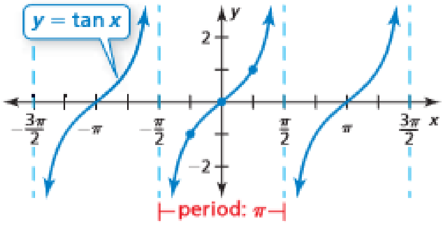
← x approaches $-\frac{\pi}{2}$ —— | —— x approaches $\frac{\pi}{2}$ →

x	$-\frac{\pi}{2}$	-1.57	-1.5	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	1.5	1.57	$\frac{\pi}{2}$
$y = \tan x$	Undef.	-1256	-14.10	-1	0	1	14.10	1256	Undef.

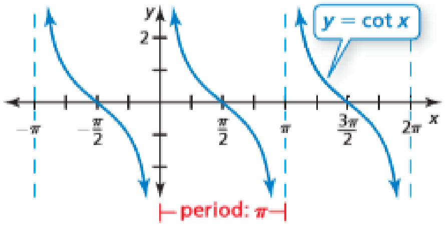
← $\tan x$ approaches $-\infty$ —— | —— $\tan x$ approaches ∞ →

Because $\tan x = \frac{\sin x}{\cos x}$, $\tan x$ is undefined for x-values at which $\cos x = 0$, such as $x = \pm\frac{\pi}{2} \approx \pm 1.571$.

The table indicates that the graph has asymptotes at these values. The table represents one cycle of the graph, so the period of the graph is π.



You can use a similar approach to graph $y = \cot x$. Because $\cot x = \frac{\cos x}{\sin x}$, $\cot x$ is undefined for x-values at which $\sin x = 0$, which are multiples of π. The graph has asymptotes at these values. The period of the graph is also π.



Core Concept

Characteristics of $y = \tan x$ and $y = \cot x$

The functions $y = \tan x$ and $y = \cot x$ have the following characteristics.

- The domain of $y = \tan x$ is all real numbers except odd multiples of $\frac{\pi}{2}$. At these x-values, the graph has vertical asymptotes.
- The domain of $y = \cot x$ is all real numbers except multiples of π. At these x-values, the graph has vertical asymptotes.
- The range of each function is all real numbers. So, the functions do not have maximum or minimum values, and the graphs do not have an amplitude.
- The period of each graph is π.
- The x-intercepts for $y = \tan x$ occur when $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.
- The x-intercepts for $y = \cot x$ occur when $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$.

Graphing Tangent and Cotangent Functions

The graphs of $y = a \tan bx$ and $y = a \cot bx$ represent transformations of their parent functions. The value of a indicates a vertical stretch ($a > 1$) or a vertical shrink ($0 < a < 1$). The value of b indicates a horizontal stretch ($0 < b < 1$) or a horizontal shrink ($b > 1$) and changes the period of the graph.

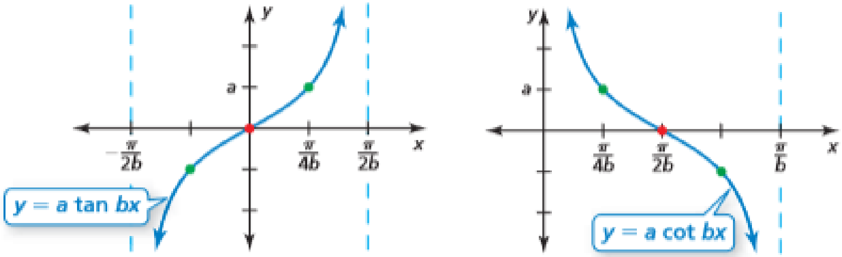
Core Concept

Period and Vertical Asymptotes of $y = a \tan bx$ and $y = a \cot bx$

The period and vertical asymptotes of the graphs of $y = a \tan bx$ and $y = a \cot bx$, where a and b are nonzero real numbers, are as follows.

- The period of the graph of each function is $\frac{\pi}{|b|}$.
- The vertical asymptotes for $y = a \tan bx$ are at odd multiples of $\frac{\pi}{2|b|}$.
- The vertical asymptotes for $y = a \cot bx$ are at multiples of $\frac{\pi}{|b|}$.

Each graph below shows five key x-values that you can use to sketch the graphs of $y = a \tan bx$ and $y = a \cot bx$ for $a > 0$ and $b > 0$. These are the x-intercept, the x-values where the asymptotes occur, and the x-values halfway between the x-intercept and the asymptotes. At each halfway point, the value of the function is either a or $-a$.



CONCEPT 1: GRAPHING TANGENT FUNCTIONS

1. Graph one period of $g(x) = 2 \tan 3x$. Describe the graph g as a transformation of the graph of $f(x) = \tan x$.

2. Graph one period of $g(x) = 3 \tan 2x$. Describe the graph g as a transformation of the graph of $f(x) = \tan x$.

9.7: Using the Trigonometric Identity (pg. 514 – 516)

Since (x, y) is on a circle centered at the origin with radius 1, it follows that the Pythagorean Theorem becomes $\boldsymbol{x^2 + y^2 = 1}$, which can be rewritten as $\mathbf{sin^2\theta + cos^2\theta = 1}$.

- You can use the quadrants of the Unit Circle to determine positive & negative sides

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

*****CONCEPT 1: FINDING TRIG VALUES*****

1. Given that $\sin\theta = \frac{4}{5}$ & $\frac{\pi}{2} < \theta < \pi$, find the two other trig functions of θ.

2. Given that $\cos\theta = -\frac{12}{13}$ & $\pi < \theta < \frac{3\pi}{2}$, find the other two trigonometric functions of θ.

3. Given that $\tan\theta = -\frac{7}{4}$ & $\frac{3\pi}{2} < \theta < 2\pi$, find the other five trigonometric functions of θ