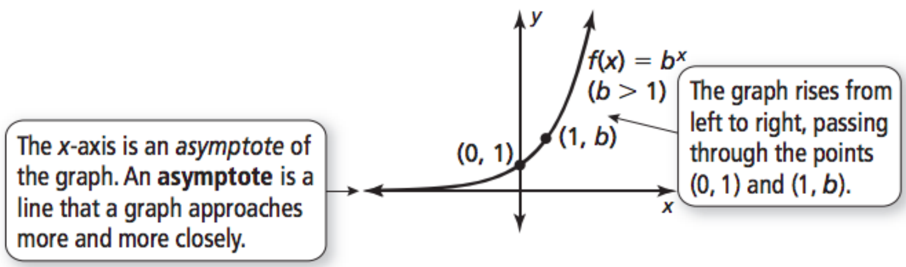
Chapter 6 Exponential and Logarithmic Functions

6.1: Exponential Growth & Decay Functions (pg. 296 – 299)

exponential function has the form $y = ab^x$.

Parent Function for Exponential Growth Functions

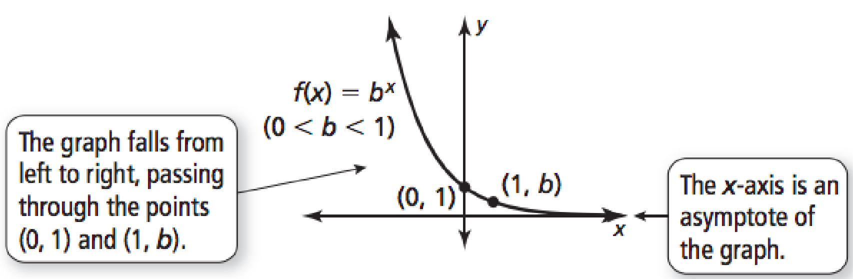
The function $f(x) = b^x$, where $b > 1$, is the parent function for the family of exponential growth functions with base b. The graph shows the general shape of an exponential growth function.



The domain of $f(x) = b^x$ is all real numbers. The range is $y > 0$.

Parent Function for Exponential Decay Functions

The function $f(x) = b^x$, where $0 < b < 1$, is the parent function for the family of exponential decay functions with base b. The graph shows the general shape of an exponential decay function.



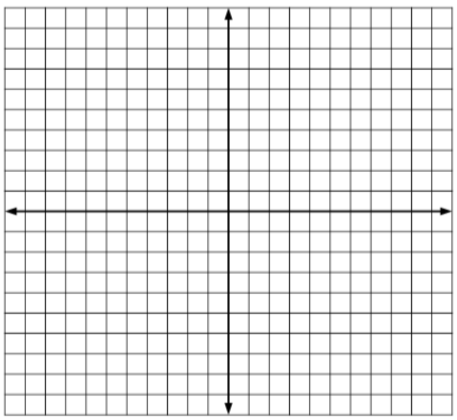
The domain of $f(x) = b^x$ is all real numbers. The range is $y > 0$.

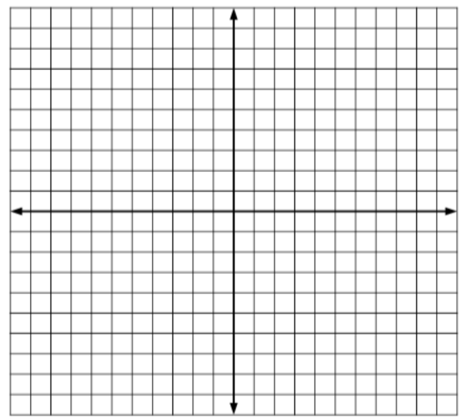
*****CONCEPT 1: GRAPHING EXPONENTIAL GROWTH & DECAY FUNCTIONS*****

1. Tell whether each function represents exponential growth (b > 1) or decay (0 < b < 1). Then graph the function.

a) $y = (2)^x$

b) $y = \left(\frac{1}{2}\right)^x$

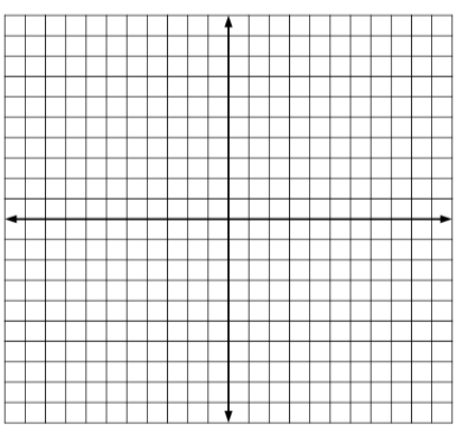


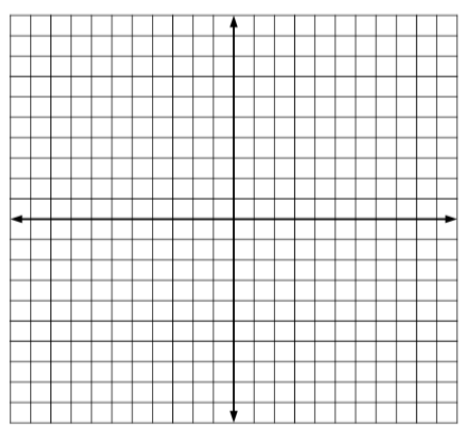


2. Tell whether each function represents exponential growth (b > 1) or decay (0 < b < 1). Then graph the function.

a) $y = (1.5)^x$

b) $y = \left(\frac{1}{3}\right)^x$





Exponential Models

Some real-life quantities increase or decrease by a fixed percent each year (or some other time period). The amount y of such a quantity after t years can be modeled by one of these equations.

Exponential Growth Model

$$y = a(1 + r)^t$$

Exponential Decay Model

$$y = a(1 - r)^t$$

Note that a is the initial amount and r is the percent increase or decrease written as a decimal. The quantity $1 + r$ is the growth factor, and $1 - r$ is the decay factor.

*****CONCEPT 2: SOLVING A REAL-LIFE PROBLEM*****

3. The value of a car *y* (in thousands of dollars) can be approximated by the model $y = 25(0.85)^t$, where *t* is the number of years since the car was new.

a) Tell whether the model represents exponential growth or exponential decay.

b) Identify the annual percent increase or decrease in the value of the car.

4. The value of a car *y* (in thousands of dollars) can be approximated by the model $y = 31(0.92)^t$, where *t* is the number of years since the car was new.

a) Tell whether the model represents exponential growth or exponential decay.

b) Identify the annual percent increase or decrease in the value of the car.

*****CONCEPT 3: WRITING AN EXPONENTIAL MODEL*****

5. In 2000, the world population was about 6.09 billion. During the next 13 years, the world population increase by about 1.18% each year.

a) Write an exponential growth model giving the population *y* (in billions) *t* years after 2000.

b) Estimate the world population in 2005.

*****CONCEPT 4: REWRITING AN EXPONENTIAL FUNCTION*****

6. The amount y (in grams) of the radioactive isotope chromium-51 remaining after t days is $y = a(0.5)^{\frac{t}{28}}$, where a is the initial amount (in grams). What percent of the chromium-51 decays each day?

7. The amount y (in grams) of the radioactive isotope barium-140 remaining after t days is $y = a(0.5)^{\frac{t}{13}}$, where a is the initial amount (in grams). What percent of the barium-140 decays each day?

Compound Interest

Consider an initial principal P deposited in an account that pays interest at an annual rate r (expressed as a decimal), compounded n times per year. The amount A in the account after t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

*****CONCEPT 5: FINDING THE BALANCE IN AN ACCOUNT*****

8. You deposit $9000 in an account that pays 1.46% annual interest. Find the balance after 3 years when the interest is compounded quarterly.

9. You deposit $8600 in an account that pays 1.32% annual interest. Find the balance after 4 years when the interest is compounded quarterly.

6.2: The Natural Base *e* (pg. 304 – 306)

The Natural Base e

The natural base e is irrational. It is defined as follows:

As x approaches $+\infty$, $\left(1 + \frac{1}{x}\right)^x$ approaches $e \approx 2.71828182846$.

*****CONCEPT 1: SIMPLIFYING NATURAL BASE EXPRESSIONS*****

1. Simplifying each expression.

a) $e^3 \cdot e^6$ b) $\frac{16e^5}{4e^4}$ c) $(3e^{-4x})^2$

2. Simplifying each expression.

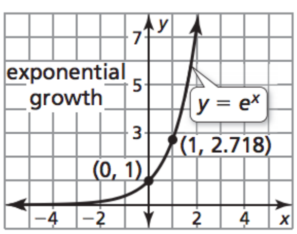
a) $e^2 \cdot e^9$ b) $\frac{25e^{13}}{5e^{12}}$ c) $(2e^{-3x})^5$

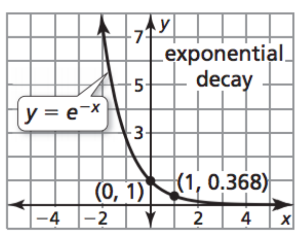
Natural Base Functions

A function of the form $y = ae^{rx}$ is called a *natural base exponential function.*

- When $a > 0$ and $r > 0$, the function is an exponential growth function.
- When $a > 0$ and $r < 0$, the function is an exponential decay function.

The graphs of the basic functions $y = e^x$ and $y = e^{-x}$ are shown.

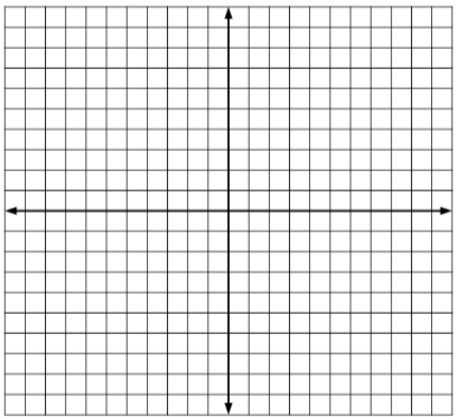




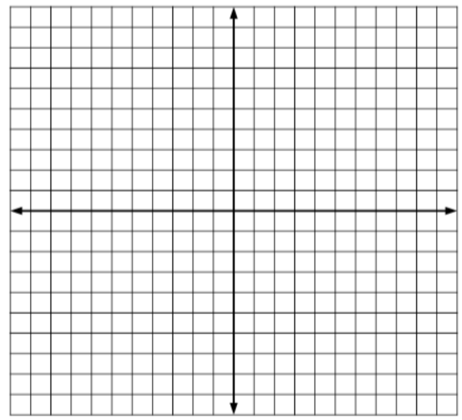
*****CONCEPT 2: GRAPHING NATURAL BASE FUNCTIONS*****

3. Tell whether each function represents exponential growth or exponential decay. Then graph the function.

a) $y = 3e^x$

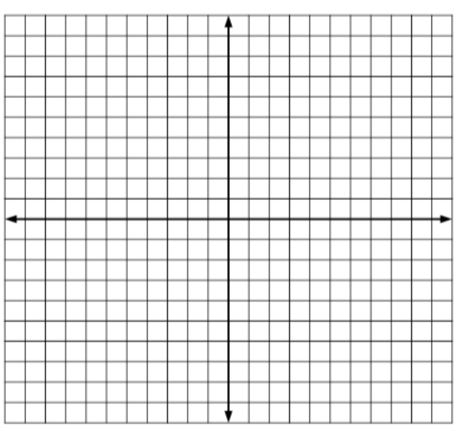


b) $f(x) = e^{-0.5x}$

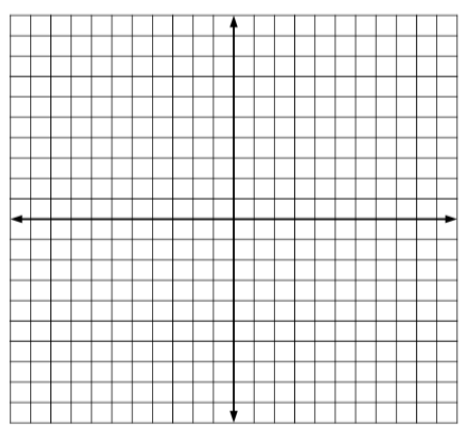


4. Tell whether each function represents exponential growth or exponential decay. Then graph the function.

a) $f(x) = 2.5e^x$



b) $y = e^{-0.2x}$



Continuously Compounded Interest

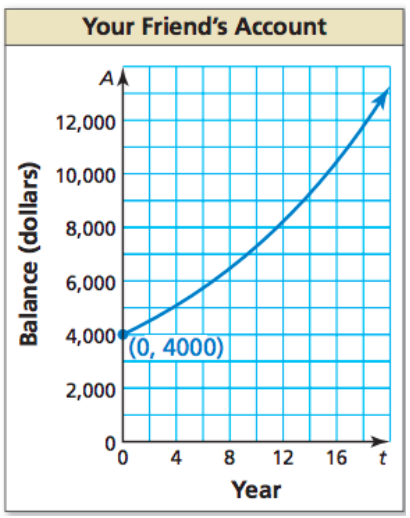
When interest is compounded *continuously*, the amount A in an account after t years is given by the formula

$$A = Pe^{rt}$$

where P is the principal and r is the annual interest rate expressed as a decimal.

*****CONCEPT 3: MODELING WITH MATHEMATICS*****

5. You and your friend each have accounts that earn annual interest compounded continuously. The balance A (in dollars) of your account after t years can be modeled by $A = 4500e^{0.04t}$. The graph shows the balance of your friend's account over time. Which account has a greater principal? Which has a greater balance after 10 years?



6. You deposit $4250 in an account that earns 5% annual interest compounded continuously. Compare the balance after 10 years with the account in Example 6.

6.3 Logarithms & Logarithmic Functions (pg. 310 – 313)

Definition of Logarithm with Base *b*

Let b and y be positive real numbers with $b \neq 1$. The **logarithm of y with base b** is denoted by $\log_b y$ and is defined as

$$\log_b y = x \qquad \text{if and only if} \qquad b^x = y.$$

The expression $\log_b y$ is read as "log base b of y."

*****CONCEPT 1: REWRITING LOGARITHMIC FUNCTIONS*****

1. Rewrite each equation in exponential form.

<u>Logarithmic Form</u> <u>Exponential Form</u>

a) $\log_2 16 = 4$

b) $\log_4 1 = 0$

c) $\log_{12} 12 = 1$

d) $\log_{\frac{1}{4}} 4 = -1$

*****CONCEPT 2: REWRITING EXPONENTIAL FUNCTIONS*****

2. Rewrite each equation in logarithmic form.

<u>Exponential Form</u> <u>Logarithmic Form</u>

a) $5^2 = 25$

b) $10^{-1} = 0.1$

c) $8^{\frac{2}{3}} = 4$

d) $6^{-3} = \frac{1}{216}$

*****CONCEPT 3: EVALUATING LOGARITHMIC EXPRESSIONS*****

HINT: To help you find the value of log (*b*)*y*, ask which power of *b* gives you *y*

3. Evaluate each logarithm.

a) $\log_4 64$

b) $\log_5 0.2$

c) $\log_{\frac{1}{5}} 125$

d) $\log_{36} 6$

*****CONCEPT 4: EVALUATING COMMON & NATURAL LOGARITHMS*****

4. Evaluate (a) $\log 8$ and (b) $\ln 0.3$ using a calculator. Round your answer to the third decimal place.

Using Inverse Properties

By the definition of a logarithm, it follows that the logarithmic function $g(x) = \log_b x$ is the inverse of the exponential function $f(x) = b^x$. This means that

$$g(f(x)) = \log_b b^x = x \qquad \text{and} \qquad f(g(x)) = b^{\log_b x} = x.$$

In other words, exponential functions and logarithmic functions "undo" each other.

*****CONCEPT 5: USING INVERSE PROPERTIES*****

5. Simplify (a) $10^{\log 4}$ and (b) $\log_5 25^x$

6. Simplify (a) $10^{\log 7}$ and (b) $\log_3 27^x$

*****CONCEPT 6: FINDING INVERSE FUNCTIONS*****

7. Find the inverse of each function

a) $f(x) = 6^x$

b) $y = \ln(x+3)$

8. Find the inverse of the function.

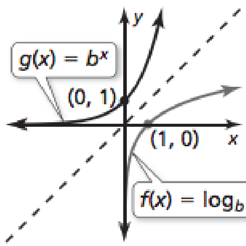
a) $f(x) = 11^x$

b) $y = \ln(x+6)$

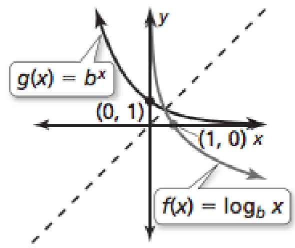
Parent Graphs for Logarithmic Functions

The graph of $f(x) = \log_b x$ is shown below for $b > 1$ and for $0 < b < 1$. Because $f(x) = \log_b x$ and $g(x) = b^x$ are inverse functions, the graph of $f(x) = \log_b x$ is the reflection of the graph of $g(x) = b^x$ in the line $y = x$.

Graph of $f(x) = \log_b x$ for $b > 1$



Graph of $f(x) = \log_b x$ for $0 < b < 1$

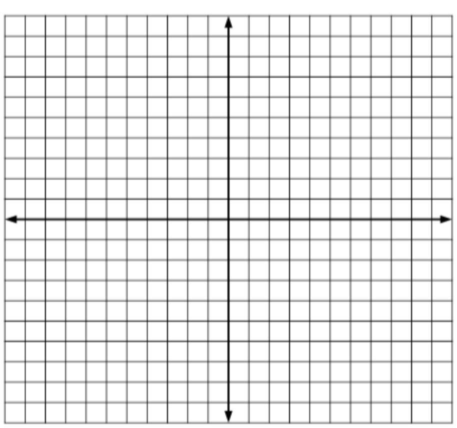


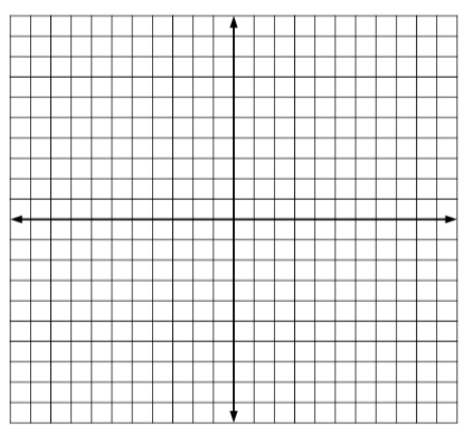
Note that the y-axis is a vertical asymptote of the graph of $f(x) = \log_b x$. The domain of $f(x) = \log_b x$ is $x > 0$, and the range is all real numbers.

*****CONCEPT 7: GRAPHING A LOGARITHMIC FUNCTION*****

9. a) Graph $f(x) = \log_3 x$

b) Graph $y = \log_7 x$



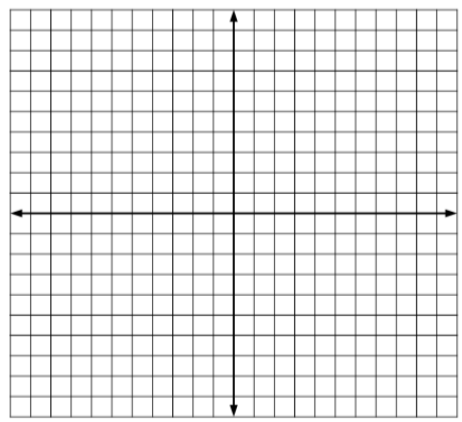


6.4 Transformations of Exponential Functions (pg. 318 – 321)

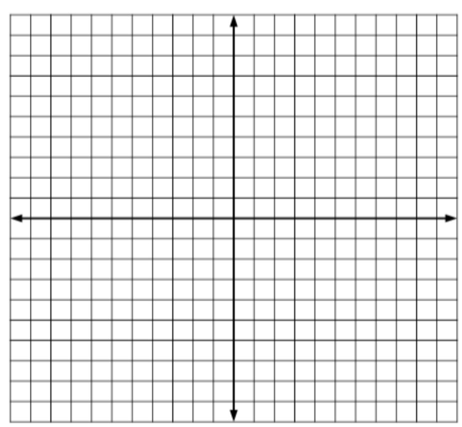
Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x-h)$	$g(x) = 4^{x-3}$ $g(x) = 4^{x+2}$	3 units right 2 units left
Vertical Translation Graph shifts up or down.	$f(x)+k$	$g(x) = 4^{x} + 5$ $g(x) = 4^{x} - 1$	5 units up 1 unit down
Reflection Graph flips over x- or y-axis.	$f(-x)$ $-f(x)$	$g(x) = 4^{-x}$ $g(x) = -4^{x}$	over y-axis over x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis	$f(ax)$	$g(x) = 4^{2x}$ $g(x) = 4^{x/2}$	shrink by $\frac{1}{2}$ stretch by 2
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis	$a \bullet f(x)$	$g(x) = 3(4^{x})$ $g(x) = \frac{1}{4}(4^{x})$	stretch by 3 shrink by $\frac{1}{4}$

*****CONCEPT 1: TRANSFORMING AN EXPONENTIAL FUNCTION*****

1. Describe the transformation of $f(x) = \left(\frac{1}{2}\right)^{x}$ represented by $g(x) = \left(\frac{1}{2}\right)^{x} - 4$. Then graph each function.

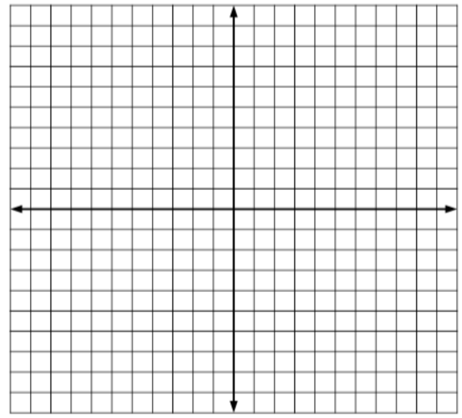


2. Describe the transformation of $f(x) = \left(\frac{1}{3}\right)^{x}$ represented by $g(x) = \left(\frac{1}{3}\right)^{x} - 5$. Then graph each function.

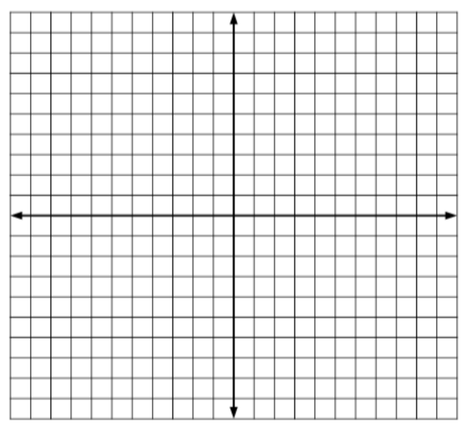


*****CONCEPT 2: TRANSLATING A NATURAL BASE EXPONENTIAL FUNCTION*****

3. Describe the transformation of $f(x) = e^x$ represented by $g(x) = e^{x+3} + 2$. Then graph each function.



4. Describe the transformation of $f(x) = e^x$ represented by $g(x) = e^{x+2} - 1$. Then graph each function.

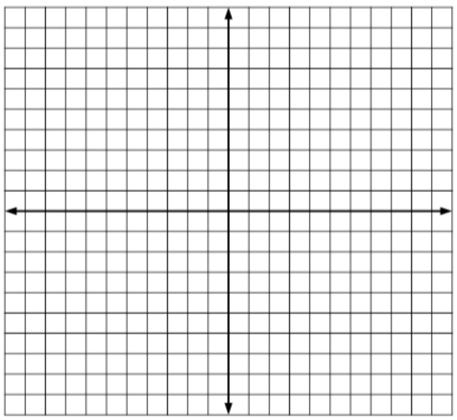


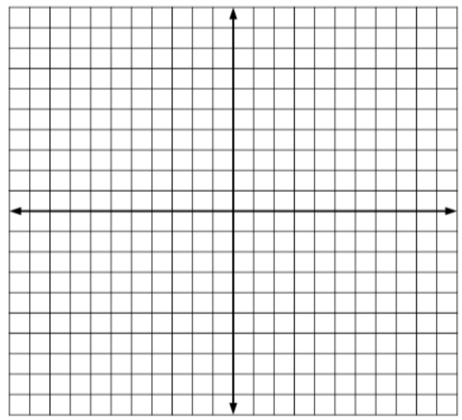
*****CONCEPT 3: TRANSFORMING EXPONENTIAL FUNCTIONS*****

5. Describe the transformation of f represented by g. Then graph each function.

a) $f(x) = 3^x, g(x) = 3^{3x-5}$

b) $f(x) = e^{-x}, g(x) = -\frac{1}{8}e^{-x}$

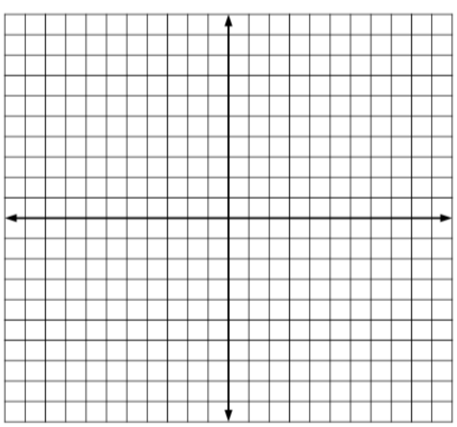


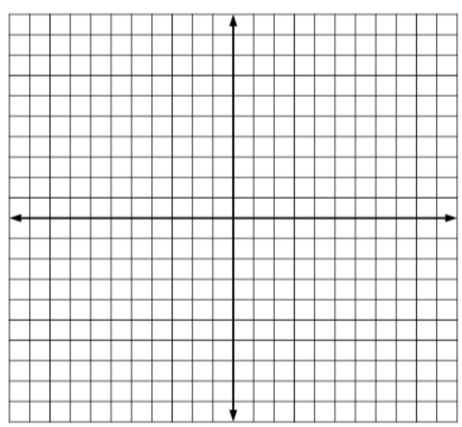


6. Describe the transformation of f represented by g. Then graph each function.

a) $f(x) = 2^x, g(x) = 2^{2x-4}$

b) $f(x) = e^{-x}, g(x) = -\frac{1}{5}e^{-x}$





Transformations of Logarithmic Functions

Transformation	**$f(x)$ Notation**	**Examples**	
Horizontal Translation Graph shifts left or right.	$f(x-h)$	$g(x) = \log(x-4)$ $g(x) = \log(x+7)$	4 units right 7 units left
Vertical Translation Graph shifts up or down.	$f(x)+k$	$g(x) = \log x + 3$ $g(x) = \log x - 1$	3 units up 1 unit down
Reflection Graph flips over x- or y-axis.	$f(-x)$ $-f(x)$	$g(x) = \log(-x)$ $g(x) = -\log x$	over y-axis over x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis	$f(ax)$	$g(x) = \log(4x)$ $g(x) = \log\left(\frac{1}{3}x\right)$	shrink by $\frac{1}{4}$ stretch by 3
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis	$a \bullet f(x)$	$g(x) = 5\log x$ $g(x) = \frac{2}{3}\log x$	stretch by 5 shrink by $\frac{2}{3}$

*****CONCEPT 4: TRANSFORMING LOGARITHMIC FUNCTIONS*****

7. Describe the transformation of f represented by g. Then graph each function.

a) $f(x) = \log x, g(x) = \log\left(-\frac{1}{2}x\right)$

b) $f(x) = \log_{\frac{1}{2}} x, g(x) = 2\log_{\frac{1}{2}}(x+4)$

8. Describe the transformation of f represented by g. Then graph each function.

a) $f(x) = \log x, g(x) = \log\left(-\frac{1}{4}x\right)$

b) $f(x) = \log_{\frac{1}{3}} x, g(x) = 3\log_{\frac{1}{3}}(x-2)$

*****CONCEPT 5: WRITING A TRANSFORMED EXPONENTIAL FUNCTION*****

9. Let the graph of g be a reflection in the x-axis followed by a translation 4 units right of the graph of $f(x) = 2^x$. Write a rule for g.

10. Let the graph of g be a reflection in the x-axis followed by a translation 2 units left of the graph of $f(x) = 3^x$. Write a rule for g.

*****CONCEPT 6: WRITING A TRANSFORMED LOGARITHMIC FUNCTION*****

11. Let the graph of g be a translation 2 units up followed by a vertical stretch by a factor of 2 of the graph of $f(x) = \log_{\frac{1}{3}} x$. Write a rule for g.

12. Let the graph of g be a translation 1 units down followed by a vertical stretch by a factor of 5 of the graph of $f(x) = \log_{\frac{1}{6}} x$. Write a rule for g.

6.5 Properties of Logarithms (pg. 328 – 330)

Properties of Logarithms

Let b, m, and n be positive real numbers with $b \neq 1$.

Product Property $\log_b mn = \log_b m + \log_b n$

Quotient Property $\log_b \frac{m}{n} = \log_b m - \log_b n$

Power Property $\log_b m^n = n \log_b m$

*****CONCEPT 1: USING PROPERTIES OF LOGS*****

1. Using $\log_2 3 \approx 1.585$ and $\log_2 7 \approx 2.807$ to evaluate each logarithm.

a) $\log_2 \frac{3}{7}$ b) $\log_2 21$ c) $\log_2 49$

2. Using $\log_3 4 \approx 1.262$ and $\log_3 5 \approx 1.465$ to evaluate each logarithm.

a) $\log_3 \frac{4}{5}$ b) $\log_3 20$ c) $\log_3 25$

*****CONCEPT 2: EXPANDING A LOGARITHMIC EXPRESSION*****

3. Expand $\ln \frac{5x^7}{y}$

4. Expand $\ln \frac{3x^5}{y}$

*****CONCEPT 3: CONDENSING A LOGARITHMIC EXPRESSION*****

5. $\log 9 + 3\log 2 - \log 3$

6. $\log 6 + 4\log 3 - \log 3$

Change-of-Base Formula

If *a, b,* and *c* are positive real numbers with $b \neq 1$ and $c \neq 1$, then

$$\log_c a = \frac{\log_b a}{\log_b c}.$$

In particular, $\log_c a = \dfrac{\log a}{\log c}$ and $\log_c a = \dfrac{\ln a}{\ln c}$.

*****CONCEPT 4: CHANGING A BASE USING COMMON LOGARITHMS*****

7. Evaluate $\log_3 8$ using common logarithms.

8. Evaluate $\log_7 11$ using common logarithms.

*****CONCEPT 5: CHANGING A BASE USING NATURAL LOGARITHMS*****

9. Evaluate $\log_6 24$ using natural logarithms.

10. Evaluate $\log_2 19$ using natural logarithms.

6.6: Solving Exponential & Logarithmic Equations (pg. 334 – 337)

Property of Equality for Exponential Equations

Algebra If b is a positive real number other than 1, then $b^x = b^y$ if and only if $x = y$.

Example If $3^x = 3^5$, then $x = 5$. If $x = 5$, then $3^x = 3^5$.

*****CONCEPT 1: SOLVING EXPONENTIAL EQUATIONS*****

1. Solve each equation.

a) $100^x = \left(\frac{1}{10}\right)^{x-3}$

b) $2^x = 7$

2. a) $81^x = \left(\frac{1}{9}\right)^{x+3}$

b) $3^x = 8$

Property of Equality for Logarithmic Equations

Algebra If b, x, and y are positive real numbers with $b \neq 1$, then $\log_b x = \log_b y$ if and only if $x = y$.

Example If $\log_2 x = \log_2 7$, then $x = 7$. If $x = 7$, then $\log_2 x = \log_2 7$.

*****CONCEPT 2: SOLVING LOGARITHMIC EQUATIONS*****

3. Solve (a) $\ln(4x - 7) = \ln(x + 5)$ and (b) $\log_2(5x - 17) = 3$

4. a) $\ln(5x + 2) = \ln(x + 10)$

b) $\log_3(10x - 9) = 4$

*****CONCEPT 3: SOLVING A LOGARITHMIC EQUATION*****

5. Solve $\log 2x + \log(x - 5) = 2$

6. Solve $\log_6 6x + \log_6(x - 1) = 2$

7. Solve $\log_4(x + 12) + \log_4 x = 3$

6.7: Modeling with Exponential Functions (pg. 342 – 345)

Vocabulary:
Finite Differences: The differences of consecutive y-values in a data set when the x-values are equally spaced.
Common Ratio: The constant ratio r between consecutive terms of a geometric sequence.

*****CONCEPT 1: CLASSIFYING DATA SETS*****

(ask yourself: Are the x-values equally spaced? If so, look for pattern in y-values)

1. Determine the type of functions represented by each table.

a.

x	−2	−1	0	1	2	3	4
y	0.5	1	2	4	8	16	32

b.

x	−2	0	2	4	6	8	10
y	2	0	2	8	18	32	50

2. Determine the type of functions represented by each table.

a.

x	−2	−1	0	1	2	3
y	243	81	27	9	3	1

b.

x	−2	0	2	4	6	8
y	13	5	13	37	77	133

*****CONCEPT 2: WRITING AN EXPONENTIAL FUNCTION USING TWO POINTS*****

3. Write an exponential function $y = ab^x$ whose graph passes through $(1, 6)$ & $(3, 54)$.

4. Write an exponential function $y = ab^x$ whose graph passes through $(1, 10)$ & $(3, 40)$.

*****CONCEPT 3: FINDING AN EXPONENTIAL MODEL*****

5. A store sells trampolines. The table shows the number y of trampolines sold during the xth year that the store has been open. Write a function that models the data.

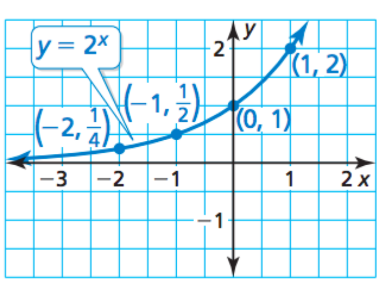
Year, x	Number of trampolines, y
1	12
2	16
3	25
4	36
5	50
6	67
7	96

6. A store sells basketball hoops. The table shows the number y of hoops sold during the xth year that the store has been open. Write a function that models the data.

Year, x	Number of hoops, y
1	17
2	31
3	50
4	85
5	156
6	274
7	498

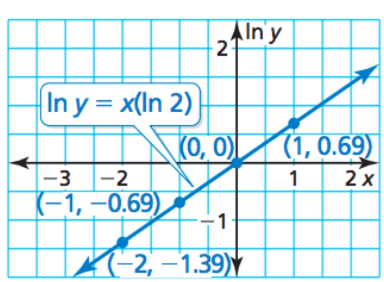
A set of more than two points (x, y) fits an exponential pattern if and only if the set of transformed points $(x, \ln y)$ fits a linear pattern.

Graph of points (x, y)



The graph is an exponential curve.

Graph of points $(x, \ln y)$



The graph is a line.

*****CONCEPT 4: WRITING A TRANSFORMED MODEL USING TRANSFORMED POINTS*****

7. Use the data form Example 5. Create a scatter plot of the data pairs $(x, \ln y)$ to show that an exponential model should be a good fit for the original data pairs (x, y). Then write an exponential model for the original data.

Year, x	Number of trampolines, y
1	12
2	16
3	25
4	36
5	50
6	67
7	96

8. Use the data form Example 6. Create a scatter plot of the data pairs $(x, \ln y)$ to show that an exponential model should be a good fit for the original data pairs (x, y). Then write an exponential model for the original data.

Year, x	Number of hoops, y
1	17
2	31
3	50
4	85
5	156
6	274
7	498