Ch. 5 Rational Exponents & Radical Functions

5.0 Properties of Exponents and Radicals (not in the algebra II textbook)

Core Concept

Zero Exponent

Words For any nonzero number a, $a^0 = 1$. The power 0^0 is undefined.

Numbers $4^0 = 1$ **Algebra** $a^0 = 1$, where $a \neq 0$

Negative Exponents

Words For any integer n and any nonzero number a, a^{-n} is the reciprocal of a^n.

Numbers $4^{-2} = \frac{1}{4^2}$ **Algebra** $a^{-n} = \frac{1}{a^n}$, where $a \neq 0$

*****CONCEPT 1: USING ZERO & NEGATIVE EXPONENTS*****

1. Evaluate each expression. a) 6^0 b) $(2)^{-4}$ c) $\frac{-5^0}{2^{-2}}$

2. Evaluate each expression. a) $(-9)^0$ b) $(-3)^{-3}$ c) $\frac{3^{-2}x^{-5}}{y^0}$.

Core Concept

Product of Powers Property

Let a be a real number, and let m and n be integers.

Words To multiply powers with the same base, add their exponents.

Numbers $4^6 \cdot 4^3 = 4^{6+3} = 4^9$ **Algebra** $a^m \cdot a^n = a^{m+n}$

Quotient of Powers Property

Let a be a nonzero real number, and let m and n be integers.

Words To divide powers with the same base, subtract their exponents.

Numbers $\frac{4^6}{4^3} = 4^{6-3} = 4^3$ **Algebra** $\frac{a^m}{a^n} = a^{m-n}$, where $a \neq 0$

Power of a Power Property

Let a be a real number, and let m and n be integers.

Words To find a power of a power, multiply the exponents.

Numbers $(4^6)^3 = 4^{6 \cdot 3} = 4^{18}$ **Algebra** $(a^m)^n = a^{mn}$

*****CONCEPT 2: USING PROPERTIES OF EXPONENTS*****

3. Simplify each expression. Write your answer using only positive exponents.

a) $3^2 \cdot 3^6$ b) $\frac{(-4)^2}{(-4)^7}$ c) $(z^4)^{-3}$

4. Simplify each expression. Write your answer using only positive exponents.

a) $10^4 \cdot 10^{-6}$

b) $\frac{(-5)^8}{(-5)^4}$

c) $x^9 \cdot x^{-9}$

d) $\frac{y^6}{y^7}$

e) $(6^{-2})^{-1}$

f) $(z^{12})^5$

Core Concept

Power of a Product Property

Let a and b be real numbers, and let m be an integer.

Words To find a power of a product, find the power of each factor and multiply.

Numbers $(3 \cdot 2)^5 = 3^5 \cdot 2^5$ **Algebra** $(ab)^m = a^m b^m$

Power of a Quotient Property

Let a and b be real numbers with $b \neq 0$, and let m be an integer.

Words To find the power of a quotient, find the power of the numerator and the power of the denominator and divide.

Numbers $\left(\frac{3}{2}\right)^5 = \frac{3^5}{2^5}$ **Algebra** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, where $b \neq 0$

*****CONCEPT 3: USING PROPERTIES OF EXPONENTS *****

5. Simplify each expression. Write your answer using only positive exponents.

a) $(-1.5y)^2$

b) $\left(\frac{a}{-10}\right)^3$

c) $\left(\frac{3d}{2}\right)^4$

d) $\left(\frac{2x}{3}\right)^{-5}$

6. Simplify each expression. Write your answer using only positive exponents.

a) $(10y)^{-3}$

b) $\left(\frac{4}{n}\right)^5$

c) $\left(\frac{1}{2k^2}\right)^4$

d) $\left(\frac{6c}{7}\right)^{-2}$

5.1: nth Roots and Rational Exponents (pg. 238 - 240)

$\sqrt[n]{a} = a^{1/n}$ Definition of rational exponent

Core Concept

Real *n*th Roots of *a*

Let n be an integer ($n > 1$) and let a be a real number.

***n* is an even integer.**		***n* is an odd integer.**	
$a < 0$	No real nth roots	$a < 0$	One real nth root: $\sqrt[n]{a} = a^{1/n}$
$a = 0$	One real nth root: $\sqrt[n]{0} = 0$	$a = 0$	One real nth root: $\sqrt[n]{0} = 0$
$a > 0$	Two real nth roots: $\pm\sqrt[n]{a} = \pm a^{1/n}$	$a > 0$	One real nth root: $\sqrt[n]{a} = a^{1/n}$

*****CONCEPT 1: FINDING Nth ROOTS*****

Find the indicated real nth root(s) of a.

1. a) $n = 3, a = -216$ b) $n = 4, a = 81$

2. a) $n = 5, a = -243$ b) $n = 8, a = 256$

*****CONCEPT 2: PRODUCTS OF SQUARE ROOTS*****

Core Concept

Product Property of Square Roots

Words The square root of a product equals the product of the square roots of the factors.

Numbers $\sqrt{9 \cdot 5} = \sqrt{9} \cdot \sqrt{5} = 3\sqrt{5}$

Algebra $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, where $a, b \geq 0$

3. Simplify the expression.

a) $\sqrt{24}$ b) $-\sqrt{80}$ c) $\sqrt{49x^3}$ d) $\sqrt{75x^5}$

4. Simplify the expression.

a) $\sqrt{96}$ b) $-\sqrt{50}$ c) $\sqrt{12x^4}$ d) $\sqrt{\frac{4x^2}{64}}$

e) $\sqrt[3]{54}$ f) $\sqrt[3]{16x^4}$ g) $\sqrt[3]{\frac{a}{-27}}$ h) $\sqrt[3]{\frac{25c^7d^3}{64}}$

Core Concept

Rational Exponents

Let $a^{1/n}$ be an nth root of a, and let m be a positive integer.

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$$

$$a^{-m/n} = \frac{1}{a^{m/n}} = \frac{1}{(a^{1/n})^m} = \frac{1}{(\sqrt[n]{a})^m},\ a \neq 0$$

*****CONCEPT 3: EVALUATING EXPRESSIONS WITH RATIONAL EXPONENTS*****

5. Evaluate each expression ***without*** using a calculator.

a) $16^{\frac{3}{2}}$ b) $32^{\frac{-3}{5}}$

6. a) $64^{\frac{4}{3}}$

b) $16^{\frac{-5}{4}}$

*****CONCEPT 3: SOLVING EQUATIONS USING Nth ROOTS*****

7. Find the real solutions

a) $4x^5 = 128$

b) $(x-3)^4 = 21$

8. a) $5x^3 = 320$

b) $(x+3)^4 = 24$

5.2: Properties of Rational Exponents and Radicals (pg. 244 - 247)

Core Concept

Properties of Rational Exponents

Let a and b be real numbers and let m and n be rational numbers, such that the quantities in each property are real numbers.

Property Name	Definition	Example
Product of Powers	$a^m \cdot a^n = a^{m+n}$	$5^{1/2} \cdot 5^{3/2} = 5^{(1/2+3/2)} = 5^2 = 25$
Power of a Power	$(a^m)^n = a^{mn}$	$(3^{5/2})^2 = 3^{(5/2 \cdot 2)} = 3^5 = 243$
Power of a Product	$(ab)^m = a^m b^m$	$(16 \cdot 9)^{1/2} = 16^{1/2} \cdot 9^{1/2} = 4 \cdot 3 = 12$
Negative Exponent	$a^{-m} = \frac{1}{a^m}, a \neq 0$	$36^{-1/2} = \frac{1}{36^{1/2}} = \frac{1}{6}$
Zero Exponent	$a^0 = 1, a \neq 0$	$213^0 = 1$
Quotient of Powers	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$	$\frac{4^{5/2}}{4^{1/2}} = 4^{(5/2-1/2)} = 4^2 = 16$
Power of a Quotient	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$	$\left(\frac{27}{64}\right)^{1/3} = \frac{27^{1/3}}{64^{1/3}} = \frac{3}{4}$

*****CONCEPT 1: USING PROPERTIES OF EXPONENTS*****

1. Use the properties of rational exponents to simplify the expression.

a) $7^{\frac{1}{4}} \cdot 7^{\frac{1}{2}}$

b) $\left(6^{\frac{1}{2}} \cdot 4^{\frac{1}{3}}\right)^2$

c) $(4^5 \cdot 3^5)^{-\frac{1}{5}}$

d) $\frac{125}{125^{\frac{1}{3}}}$

e) $\left(\frac{48^{\frac{1}{3}}}{6^{\frac{1}{3}}}\right)^2$

2. Use the properties of rational exponents to simplify the expression.

a) $2^{\frac{3}{4}} \cdot 2^{\frac{1}{2}}$

b) $\left(5^{\frac{1}{3}} \cdot 7^{\frac{1}{4}}\right)^3$

c) $(2^4 \cdot 8^4)^{-\frac{1}{4}}$

d) $\frac{3}{3^{\frac{1}{4}}}$

e) $\left(\frac{20^{\frac{1}{2}}}{5^{\frac{1}{2}}}\right)^3$

Core Concept

Properties of Radicals

Let a and b be real numbers and let n be an integer greater than 1.

Property Name	Definition	Example
Product Property	$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$	$\sqrt[3]{4} \cdot \sqrt[3]{2} = \sqrt[3]{8} = 2$
Quotient Property	$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, b \neq 0$	$\frac{\sqrt[4]{162}}{\sqrt[4]{2}} = \sqrt[4]{\frac{162}{2}} = \sqrt[4]{81} = 3$

*****CONCEPT 2: USING PROPERTIES OF RADICALS*****

3. Use the properties of rational exponents to simplify the expression.

a) $\sqrt[3]{12} \cdot \sqrt[3]{18}$

b) $\frac{\sqrt[4]{80}}{\sqrt[4]{5}}$

4. a) $\sqrt[3]{4} \cdot \sqrt[3]{128}$

b) $\frac{\sqrt[5]{192}}{\sqrt[5]{6}}$

*****CONCEPT 3: RADICALS IN SIMPLEST FORM*****

5. Write the expression in simplest form.

a) $\sqrt[3]{135}$

b) $\frac{\sqrt[5]{7}}{\sqrt[5]{8}}$

6. a) $\sqrt[3]{500}$

b) $\frac{\sqrt[4]{19}}{\sqrt[4]{4}}$

*****CONCEPT 4: WRITING A RADICAL EXPRESSION IN SIMPLEST FORM*****

7. Write $\frac{2}{5+\sqrt{3}}$ in simplest form.

8. Write $\frac{3}{\sqrt{7}-2}$ in simplest form.

*****CONCEPT 5: ADDING AND SUBTRACTING LIKE RADICALS AND ROOTS*****

9. Simplify each expression.

a) $\sqrt[4]{10} + 7\sqrt[4]{10}$ b) $2\left(8^{\frac{1}{5}}\right) + 10\left(8^{\frac{1}{5}}\right)$ c) $\sqrt[3]{54} - \sqrt[3]{2}$

10.a) $\sqrt[6]{17} + 9\sqrt[6]{17}$ b) $5\left(7^{\frac{1}{3}}\right) + 6\left(7^{\frac{1}{3}}\right)$ c) $\sqrt[4]{48} - \sqrt[4]{3}$

*****CONCEPT 6: SIMPLIFYING VARIABLE EXPRESSIONS*****

11. Simplify each expression.

a) $\sqrt[3]{64y^6}$ b) $\sqrt[4]{\frac{x^4}{y^8}}$

12. a) $\sqrt[3]{27y^{12}}$ b) $\sqrt[4]{\frac{x^4}{y^{16}}}$

CONCEPT 7: WRITING VARIABLE EXPRESSIONS IN SIMPLEST FORM

13. Write each expression in simplest form. Assume all variables are positive.

a) $\sqrt[5]{4a^8b^{14}c^5}$ b) $\frac{x}{\sqrt[3]{y^8}}$ c) $\frac{14x^2y^{-\frac{1}{3}}}{2x^{\frac{3}{4}}z^{\frac{1}{6}}}$

14. a) $\sqrt[3]{2a^{13}b^3c^8}$ b) $\frac{x^2}{\sqrt[3]{y^{13}}}$ c) $\frac{16xy^{\frac{1}{3}}z}{8x^{\frac{2}{3}}y^{\frac{5}{6}}z^{-4}}$

CONCEPT 8: ADDING AND SUBTRACTING VARIABLE EXPRESSIONS

15. Perform each indicated operation. Assume all variables are positive.

a) $5\sqrt{y} + 6\sqrt{y}$ b) $12\sqrt[3]{2z^5} - z\sqrt[3]{54z^2}$

16. Perform each indicated operation. Assume all variables are positive.

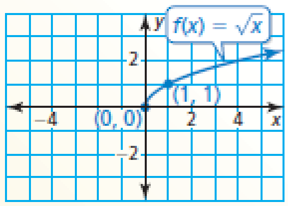
a) $6\sqrt[3]{x} + 2\sqrt[3]{x}$ b) $16\sqrt[4]{3z^6} - z\sqrt[4]{48z^2}$

5.3: Graphing Radical Functions (pg. 252 - 255)

Core Concept

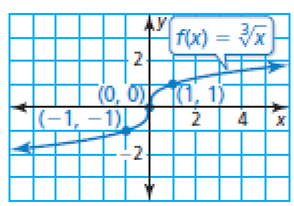
Parent Functions for Square Root and Cube Root Functions

The parent function for the family of square root functions is $f(x) = \sqrt{x}$.



Domain: $x \geq 0$, Range: $y \geq 0$

The parent function for the family of cube root functions is $f(x) = \sqrt[3]{x}$.

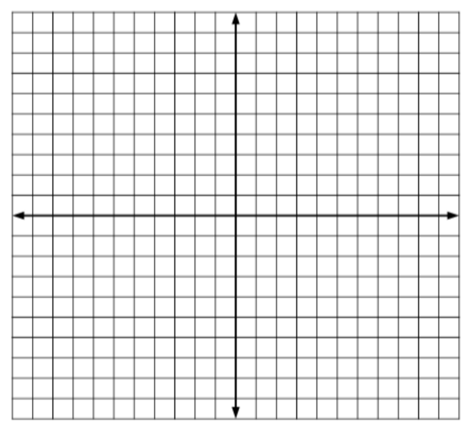


Domain and range: All real numbers

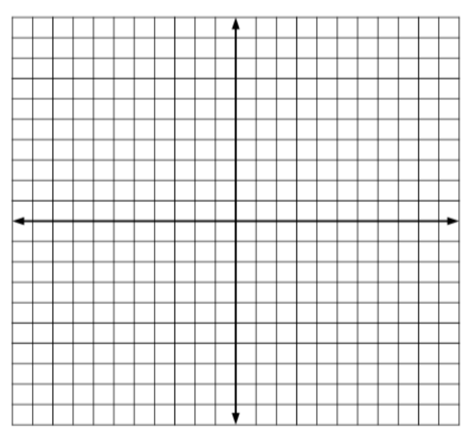
*****CONCEPT 1: GRAPHING RADICAL FUNCTIONS*****

1. Graph each function. Identify the domain and range of each function.
(Hint: Make a table of five values & sketch the graph)

$f(x) = \sqrt{\frac{1}{4}x}$

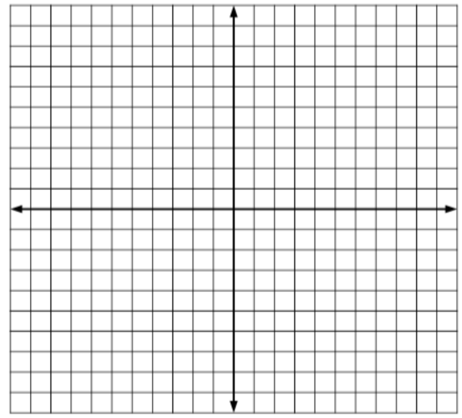


2. $g(x) = -3\sqrt[3]{x}$

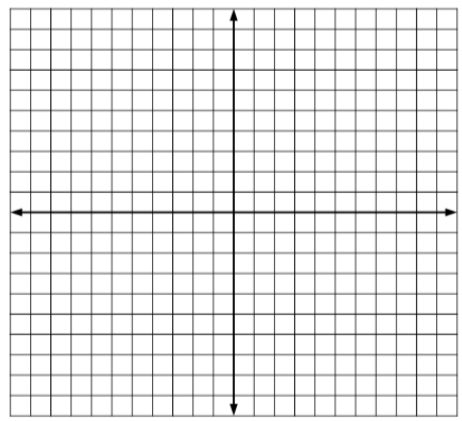


#3-4, graph each function. Then, identify the domain and range.

3. $f(x) = \sqrt{\frac{1}{3}x}$



4. $g(x) = -2\sqrt[3]{x}$



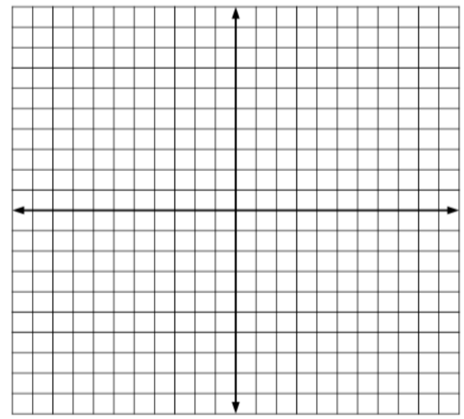
Core Concept

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x-h)$	$g(x)=\sqrt{x-2}$ $g(x)=\sqrt{x+3}$	2 units right 3 units left
Vertical Translation Graph shifts up or down.	$f(x)+k$	$g(x)=\sqrt{x}+7$ $g(x)=\sqrt{x}-1$	7 units up 1 unit down
Reflection Graph flips over x- or y-axis.	$f(-x)$ $-f(x)$	$g(x)=\sqrt{-x}$ $g(x)=-\sqrt{x}$	in the y-axis in the x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis.	$f(ax)$	$g(x)=\sqrt{3x}$ $g(x)=\sqrt{\frac{1}{2}x}$	shrink by a factor of $\frac{1}{3}$ stretch by a factor of 2
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis.	$a \cdot f(x)$	$g(x)=4\sqrt{x}$ $g(x)=\frac{1}{5}\sqrt{x}$	stretch by a factor of 4 shrink by a factor of $\frac{1}{5}$

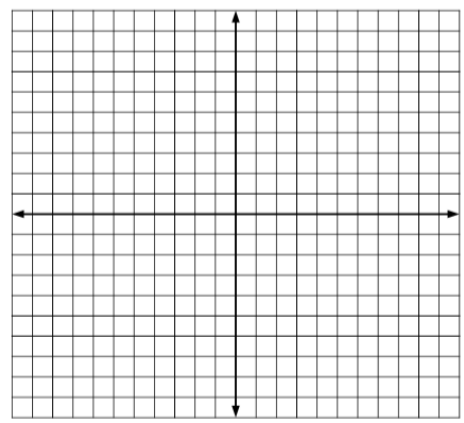
*****CONCEPT 2: TRANSFORMING RADICAL FUNCTIONS*****

5. Describe the transformation of *f* represented by *g*. Then graph each function.

a) $f(x) = \sqrt{x},\ g(x) = \sqrt{x-3} + 4$

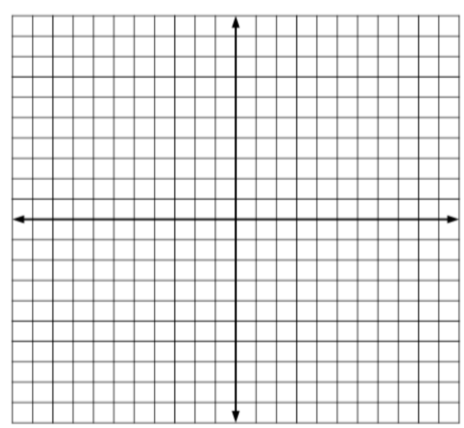


b) $f(x) = \sqrt[3]{x},\ g(x) = \sqrt[3]{-8x}$

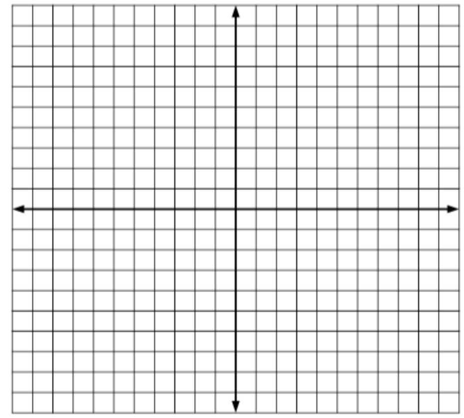


6. Describe the transformation of *f* represented by *g*. Then graph each function.

a) $f(x) = \sqrt{x},\ g(x) = \sqrt{x+2} - 3$



b) $f(x) = \sqrt[3]{x},\ g(x) = -\sqrt[3]{2x}$



*****CONCEPT 3: WRITING A TRANSFORMED RADICAL FUNCTION*****

7. Let the graph be a horizontal shrink by a factor of $\frac{1}{6}$ followed by a translation 3 units to the left of the graph of $f(x) = \sqrt[3]{x}$. Write the rule for g.

8. Let the graph be a horizontal stretch by a factor of 3 followed by a translation 6 units to the right of the graph of $f(x) = \sqrt[3]{x}$. Write the rule for g.

5.4: Solving Radical Equations and Inequalities (pg. 262 - 265)

Core Concept

Solving Radical Equations

To solve a radical equation, follow these steps:

Step 1 Isolate the radical on one side of the equation, if necessary.

Step 2 Raise each side of the equation to the same exponent to eliminate the radical and obtain a linear, quadratic, or other polynomial equation.

Step 3 Solve the resulting equation using techniques you learned in previous chapters. Check your solution.

*****CONCEPT 1: SOLVING RADICAL EQUATIONS*****

1. Solve (a) $2\sqrt{x+1} = 4$ b) $\sqrt[3]{2x-9} - 1 = 2$.

2. Solve the equations.

a) $2\sqrt{x+2} = 8$ b) $\sqrt[3]{2x-5} - 2 = 3$

*****CONCEPT 2: SOLVING AN EQUATION WITH AN EXTRANEOUS SOLUTION*****

Extraneous solution: a solution that emerges from the solving of a problem but is not **valid.**

3. Solve $x + 1 = \sqrt{7x+15}$ 4. Solve $\sqrt{2x+7} = x - 4$

*****CONCEPT 3: SOLVING AN EQUATION WITH TWO RADICALS*****

5. Solve $\sqrt{x+2}+1=\sqrt{3-x}$

6. $\sqrt{x+6}-2=\sqrt{x-2}$

*****CONCEPT 4: SOLVING AN EQUATION WITH A RATIONAL EXPONENT*****

7. Solve $(2x)^{\frac{3}{4}}+2=10$

8. $(3x)^{\frac{2}{3}}-2=34$

*****CONCEPT 5: SOLVING AN EQUATION WITH A RATIONAL EXPONENT*****

10. Solve $(x+30)^{\frac{1}{2}}=x$

11. $(x+12)^{\frac{1}{2}}=x$

*****CONCEPT 6: SOLVING A RADICAL INEQUALITY*****

12. Solve $3\sqrt{x-1} \leq 12$

13. Solve $3\sqrt{x} - 4 \leq 8$

5.6: Inverse Functions (pg. 276 - 280)

Inverse Functions: Functions that *undo* each other. The inverse function is also a reflection on the line $y = x$

*****CONCEPT 1: WRITING A FORMULA FOR THE INPUT OF A FUNCTION*****

1. Let $f(x) = 2x + 3$

a) Solve $y = f(x)$ for *x*

b) Find the input when the output is -7

2. Let $f(x) = 3x - 5$

a) Solve $y = f(x)$ for *x*

b) Find the input when the output is -11

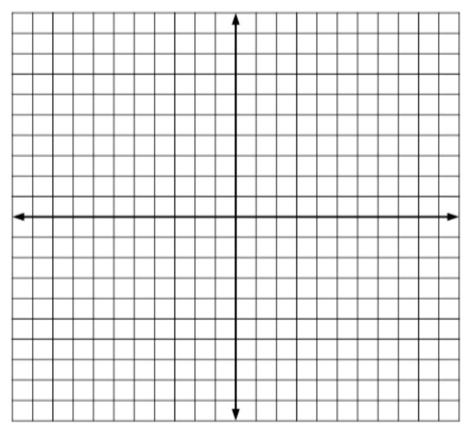
*****CONCEPT 2: FINDING THE INVERSE OF A LINEAR FUNCTION*****

3. Find the inverse of $f(x) = 3x - 1$

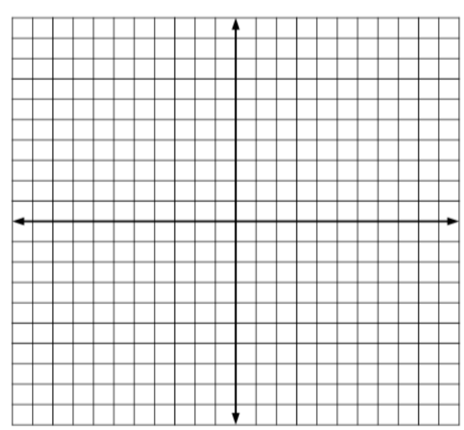
4. Find the inverse of $f(x) = 2x + 8$

*****CONCEPT 3: FINDING THE INVERSE OF A QUADRATIC FUNCTION*****

5. Find the inverse of $f(x) = x^2, x \geq 0$. Then graph the function and its inverse.
(If for the doman of f were restricted to $x \geq 0$, then the inverse range would be $y \geq 0$)



6. Find the inverse of $f(x) = x^2 - 2, x \leq 0$. Then graph the function and its inverse.
(If for the doman of f were restricted to $x \leq 0$, the inverse would be $y \leq 0 \; or \; g(x) = -\sqrt{x}$)

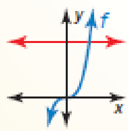


Core Concept

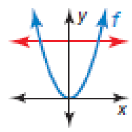
Horizontal Line Test

The inverse of a function f is also a function if and only if no horizontal line intersects the graph of f more than once.

Inverse is a function

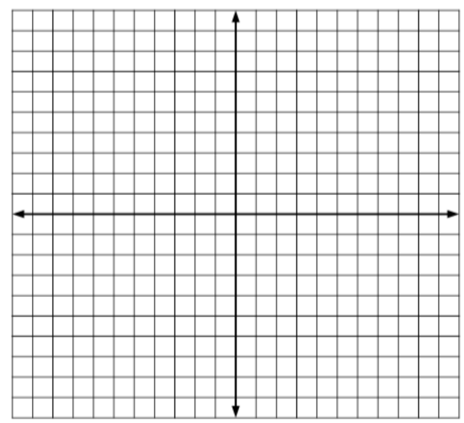


Inverse is not a function

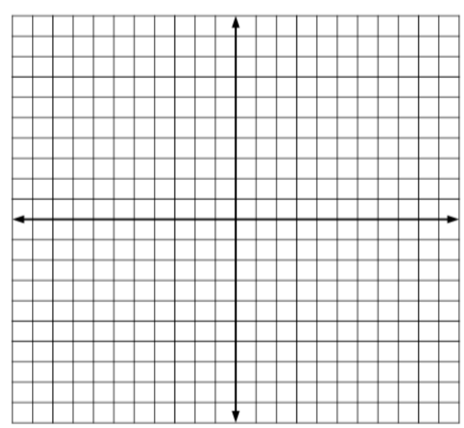


*****CONCEPT 4: FINDING THE INVERSE OF A CUBIC FUNCTION*****

10. Consider the function $f(x) = 2x^3 + 1$. Determine whether the inverse of f is a function. Then find the inverse.

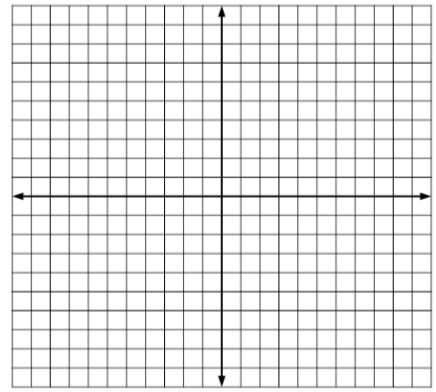


11. Consider the function $f(x) = 3x^3 - 2$. Determine whether the inverse of f is a function. Then find the inverse.

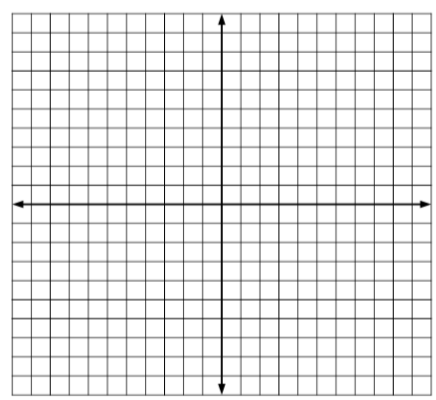


*****CONCEPT 5: FINDING THE INVERSE OF A RADICAL FUNCTION*****

12. Consider the function $f(x) = 2\sqrt{x-3}$. Determine whether the inverse of f is a function. Then find the inverse.



13. Consider the function $f(x) = 3\sqrt{x+4}$. Determine whether the inverse of f is a function. Then find the inverse.



Inverse functions undo each other. So, when you evaluate a function for a specific input, then evaluate its inverse using the output, you obtain the original input. Let *f* and *g* be inverse functions. So, in general, $f\big(g(x)\big) = x \text{ } and \text{ } g\big(f(x)\big) = x.$

*****CONCEPT 6: VERIFYING FUNCTIONS ARE INVERSES*****

14. Verify that $f(x) = 3x - 1$ and $g(x) = \frac{x+1}{3}$ are inverse functions.

15. Verify that $f(x) = 2x + 4$ and $g(x) = \frac{x-4}{2}$ are inverse functions.