# Ch. 5 Rational Exponents & Radical Functions

## 5.0 Properties of Exponents and Radicals (not in the book)

### Core Concept

**Zero Exponent**

**Words** For any nonzero number $a$, $a^0 = 1$. The power $0^0$ is undefined.

**Numbers** $4^0 = 1$ **Algebra** $a^0 = 1$, where $a \neq 0$

**Negative Exponents**

**Words** For any integer $n$ and any nonzero number $a$, $a^{-n}$ is the reciprocal of $a^n$.

**Numbers** $4^{-2} = \frac{1}{4^2}$ **Algebra** $a^{-n} = \frac{1}{a^n}$, where $a \neq 0$

*****CONCEPT 1: USING ZERO & NEGATIVE EXPONENTS*****

1. Evaluate each expression. a) $6^0$ b) $(2)^{-4}$ c) $\frac{-5^0}{2^{-2}}$

2. Evaluate each expression. a) $(-9)^0$ b) $(-3)^{-3}$ c) $\frac{3^{-2}x^{-5}}{y^0}$.

### Core Concept

**Product of Powers Property**

Let $a$ be a real number, and let $m$ and $n$ be integers.

**Words** To multiply powers with the same base, add their exponents.

**Numbers** $4^6 \cdot 4^3 = 4^{6+3} = 4^9$ **Algebra** $a^m \cdot a^n = a^{m+n}$

**Quotient of Powers Property**

Let $a$ be a nonzero real number, and let $m$ and $n$ be integers.

**Words** To divide powers with the same base, subtract their exponents.

**Numbers** $\frac{4^6}{4^3} = 4^{6-3} = 4^3$ **Algebra** $\frac{a^m}{a^n} = a^{m-n}$, where $a \neq 0$

**Power of a Power Property**

Let $a$ be a real number, and let $m$ and $n$ be integers.

**Words** To find a power of a power, multiply the exponents.

**Numbers** $(4^6)^3 = 4^{6 \cdot 3} = 4^{18}$ **Algebra** $(a^m)^n = a^{mn}$

*****CONCEPT 2: USING PROPERTIES OF EXPONENTS*****

3. Simplify each expression. Write your answer using only positive exponents.

a) $3^2 \cdot 3^6$ b) $\frac{(-4)^2}{(-4)^7}$ c) $(z^4)^{-3}$

4. Simplify each expression. Write your answer using only positive exponents.

a) $10^4 \cdot 10^{-6}$

b) $\frac{(-5)^8}{(-5)^4}$

c) $x^9 \cdot x^{-9}$

d) $\frac{y^6}{y^7}$

e) $(6^{-2})^{-1}$

f) $(z^{12})^5$

## Core Concept

**Power of a Product Property**

Let $a$ and $b$ be real numbers, and let $m$ be an integer.

**Words** To find a power of a product, find the power of each factor and multiply.

**Numbers** $(3 \cdot 2)^5 = 3^5 \cdot 2^5$ **Algebra** $(ab)^m = a^m b^m$

**Power of a Quotient Property**

Let $a$ and $b$ be real numbers with $b \neq 0$, and let $m$ be an integer.

**Words** To find the power of a quotient, find the power of the numerator and the power of the denominator and divide.

**Numbers** $\left(\frac{3}{2}\right)^5 = \frac{3^5}{2^5}$ **Algebra** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, where $b \neq 0$

*****CONCEPT 3: USING PROPERTIES OF EXPONENTS *****

5. Simplify each expression. Write your answer using only positive exponents.

a) $(-1.5y)^2$

b) $\left(\frac{a}{-10}\right)^3$

c) $\left(\frac{3d}{2}\right)^4$

d) $\left(\frac{2x}{3}\right)^{-5}$

6. Simplify each expression. Write your answer using only positive exponents.

a) $(10y)^{-3}$

b) $\left(\frac{4}{n}\right)^5$

c) $\left(\frac{1}{2k^2}\right)^4$

d) $\left(\frac{6c}{7}\right)^{-2}$

**5.1: nth Roots and Rational Exponents (pg. 238 - 240)**

$\sqrt[n]{a} = a^{1/n}$ Definition of rational exponent

## Core Concept

**Real *n*th Roots of *a***

Let $n$ be an integer ($n > 1$) and let $a$ be a real number.

| | $n$ is an even integer. | | $n$ is an odd integer. |
|---|---|---|---|
| $a < 0$ | No real $n$th roots | $a < 0$ | One real $n$th root: $\sqrt[n]{a} = a^{1/n}$ |
| $a = 0$ | One real $n$th root: $\sqrt[n]{0} = 0$ | $a = 0$ | One real $n$th root: $\sqrt[n]{0} = 0$ |
| $a > 0$ | Two real $n$th roots: $\pm\sqrt[n]{a} = \pm a^{1/n}$ | $a > 0$ | One real $n$th root: $\sqrt[n]{a} = a^{1/n}$ |

*****CONCEPT 1: FINDING Nth ROOTS*****

Find the indicated real $n$th root(s) of $a$.

1. a) $n = 3, a = -216$ b) $n = 4, a = 81$

2. a) $n = 5, a = -243$ b) $n = 8, a = 256$

*****CONCEPT 2: PRODUCTS OF SQUARE ROOTS*****

## Core Concept

**Product Property of Square Roots**

**Words** The square root of a product equals the product of the square roots of the factors.

**Numbers** $\sqrt{9 \cdot 5} = \sqrt{9} \cdot \sqrt{5} = 3\sqrt{5}$

**Algebra** $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, where $a, b \geq 0$

3. Simplify the expression.

a) $\sqrt{24}$  b) $-\sqrt{80}$  c) $\sqrt{49x^3}$  d) $\sqrt{75x^5}$

4. Simplify the expression.

a) $\sqrt{96}$  b) $-\sqrt{50}$  c) $\sqrt{12x^4}$  d) $\sqrt{\frac{4x^2}{64}}$

e) $\sqrt[3]{54}$  f) $\sqrt[3]{16x^4}$  g) $\sqrt[3]{\frac{a}{-27}}$  h) $\sqrt[3]{\frac{25c^7d^3}{64}}$

## Core Concept

**Rational Exponents**

Let $a^{1/n}$ be an $n$th root of $a$, and let $m$ be a positive integer.

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$$

$$a^{-m/n} = \frac{1}{a^{m/n}} = \frac{1}{(a^{1/n})^m} = \frac{1}{(\sqrt[n]{a})^m},\ a \neq 0$$

*****CONCEPT 3: EVALUATING EXPRESSIONS WITH RATIONAL EXPONENTS*****

5. Evaluate each expression ***without*** using a calculator.

a) $16^{\frac{3}{2}}$  b) $32^{\frac{-3}{5}}$

6. a) $64^{\frac{4}{3}}$ b) $16^{\frac{-5}{4}}$

*****CONCEPT 3: SOLVING EQUATIONS USING Nth ROOTS*****

7. Find the real solutions

a) $4x^5 = 128$ b) $(x-3)^4 = 21$

8. a) $5x^3 = 320$ b) $(x+3)^4 = 24$

**5.2: Properties of Rational Exponents and Radicals (pg. 244 - 247)**

## Core Concept

**Properties of Rational Exponents**

Let $a$ and $b$ be real numbers and let $m$ and $n$ be rational numbers, such that the quantities in each property are real numbers.

| Property Name | Definition | Example |
|---|---|---|
| Product of Powers | $a^m \cdot a^n = a^{m+n}$ | $5^{1/2} \cdot 5^{3/2} = 5^{(1/2+3/2)} = 5^2 = 25$ |
| Power of a Power | $(a^m)^n = a^{mn}$ | $(3^{5/2})^2 = 3^{(5/2 \cdot 2)} = 3^5 = 243$ |
| Power of a Product | $(ab)^m = a^m b^m$ | $(16 \cdot 9)^{1/2} = 16^{1/2} \cdot 9^{1/2} = 4 \cdot 3 = 12$ |
| Negative Exponent | $a^{-m} = \frac{1}{a^m}, a \neq 0$ | $36^{-1/2} = \frac{1}{36^{1/2}} = \frac{1}{6}$ |
| Zero Exponent | $a^0 = 1, a \neq 0$ | $213^0 = 1$ |
| Quotient of Powers | $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$ | $\frac{4^{5/2}}{4^{1/2}} = 4^{(5/2-1/2)} = 4^2 = 16$ |
| Power of a Quotient | $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$ | $\left(\frac{27}{64}\right)^{1/3} = \frac{27^{1/3}}{64^{1/3}} = \frac{3}{4}$ |

*****CONCEPT 1: USING PROPERTIES OF EXPONENTS*****

1. Use the properties of rational exponents to simplify the expression.

a) $7^{\frac{1}{4}} \cdot 7^{\frac{1}{2}}$

b) $\left(6^{\frac{1}{2}} \cdot 4^{\frac{1}{3}}\right)^2$

c) $(4^5 \cdot 3^5)^{-\frac{1}{5}}$

d) $\frac{125}{125^{\frac{1}{3}}}$

e) $\left(\frac{48^{\frac{1}{3}}}{6^{\frac{1}{3}}}\right)^2$

2. Use the properties of rational exponents to simplify the expression.

a) $2^{\frac{3}{4}} \cdot 2^{\frac{1}{2}}$

b) $\left(5^{\frac{1}{3}} \cdot 7^{\frac{1}{4}}\right)^3$

c) $(2^4 \cdot 8^4)^{-\frac{1}{4}}$

d) $\frac{3}{3^{\frac{1}{4}}}$

e) $\left(\frac{20^{\frac{1}{2}}}{5^{\frac{1}{2}}}\right)^3$

## Core Concept

### Properties of Radicals

Let $a$ and $b$ be real numbers and let $n$ be an integer greater than 1.

| Property Name | Definition | Example |
|---|---|---|
| Product Property | $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ | $\sqrt[3]{4} \cdot \sqrt[3]{2} = \sqrt[3]{8} = 2$ |
| Quotient Property | $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, b \neq 0$ | $\frac{\sqrt[4]{162}}{\sqrt[4]{2}} = \sqrt[4]{\frac{162}{2}} = \sqrt[4]{81} = 3$ |

*****CONCEPT 2: USING PROPERTIES OF RADICALS*****

3. Use the properties of rational exponents to simplify the expression.

a) $\sqrt[3]{12} \cdot \sqrt[3]{18}$

b) $\frac{\sqrt[4]{80}}{\sqrt[4]{5}}$

4. a) $\sqrt[3]{4} \cdot \sqrt[3]{128}$

b) $\frac{\sqrt[5]{192}}{\sqrt[5]{6}}$

*****CONCEPT 3: RADICALS IN SIMPLEST FORM*****

5. Write the expression in simplest form.

a) $\sqrt[3]{135}$

b) $\frac{\sqrt[5]{7}}{\sqrt[5]{8}}$

6. a) $\sqrt[3]{500}$

b) $\frac{\sqrt[4]{19}}{\sqrt[4]{4}}$

*****CONCEPT 4: WRITING A RADICAL EXPRESSION IN SIMPLEST FORM*****

7. Write $\frac{2}{5+\sqrt{3}}$ in simplest form.

8. Write $\frac{3}{\sqrt{7}-2}$ in simplest form.

*****CONCEPT 5: ADDING AND SUBTRACTING LIKE RADICALS AND ROOTS*****

9. Simplify each expression.

a) $\sqrt[4]{10} + 7\sqrt[4]{10}$ b) $2\left(8^{\frac{1}{5}}\right) + 10\left(8^{\frac{1}{5}}\right)$ c) $\sqrt[3]{54} - \sqrt[3]{2}$

10.a) $\sqrt[6]{17} + 9\sqrt[6]{17}$ b) $5\left(7^{\frac{1}{3}}\right) + 6\left(7^{\frac{1}{3}}\right)$ c) $\sqrt[4]{48} - \sqrt[4]{3}$

*****CONCEPT 6: SIMPLIFYING VARIABLE EXPRESSIONS*****

11. Simplify each expression.

a) $\sqrt[3]{64y^6}$ b) $\sqrt[4]{\frac{x^4}{y^8}}$

12. a) $\sqrt[3]{27y^{12}}$ b) $\sqrt[4]{\frac{x^4}{y^{16}}}$

*****CONCEPT 7: WRITING VARIABLE EXPRESSIONS IN SIMPLEST FORM*****

13. Write each expression in simplest form. Assume all variables are positive.

a) $\sqrt[5]{4a^8b^{14}c^5}$ b) $\frac{x}{\sqrt[3]{y^8}}$ c) $\frac{14x^2y^{-\frac{1}{3}}}{2x^{\frac{3}{4}}z^{\frac{1}{6}}}$

14. a) $\sqrt[3]{2a^{13}b^{3}c^{8}}$  b) $\frac{x^2}{\sqrt[3]{y^{13}}}$  c) $\frac{16xy^{\frac{1}{3}}z}{8x^{\frac{2}{3}}y^{\frac{5}{6}}z^{-4}}$

*****CONCEPT 8: ADDING AND SUBTRACTING VARIABLE EXPRESSIONS*****

15. Perform each indicated operation. Assume all variables are positive.

a) $5\sqrt{y} + 6\sqrt{y}$  b) $12\sqrt[3]{2z^5} - z\sqrt[3]{54z^2}$

16. Perform each indicated operation. Assume all variables are positive.

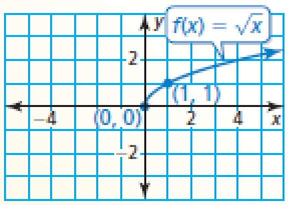
a) $6\sqrt[3]{x} + 2\sqrt[3]{x}$  b) $16\sqrt[4]{3z^6} - z\sqrt[4]{48z^2}$

**5.3: Graphing Radical Functions (pg. 252 - 255)**

## Core Concept

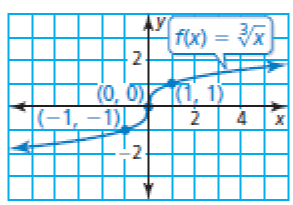
**Parent Functions for Square Root and Cube Root Functions**

The parent function for the family of square root functions is $f(x) = \sqrt{x}$.




Domain: $x \geq 0$, Range: $y \geq 0$

The parent function for the family of cube root functions is $f(x) = \sqrt[3]{x}$.

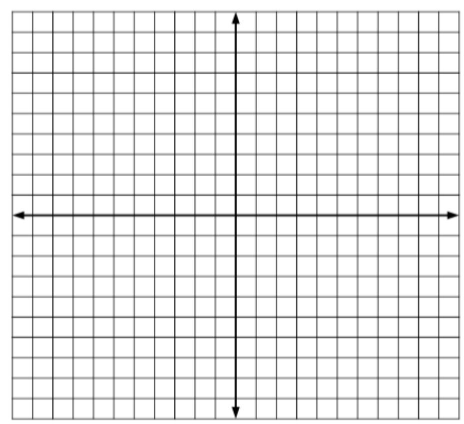



Domain and range: All real numbers

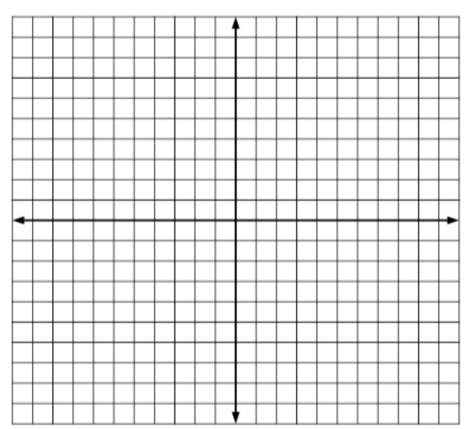
*****CONCEPT 1: GRAPHING RADICAL FUNCTIONS*****

1. Graph each function. Identify the domain and range of each function.
(Hint: Make a table of five values & sketch the graph)

$f(x) = \sqrt{\frac{1}{4}x}$

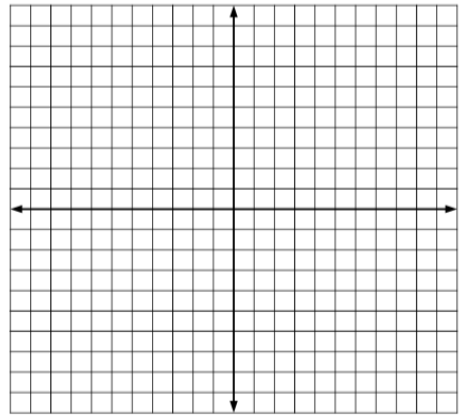


2. $g(x) = -3\sqrt[3]{x}$

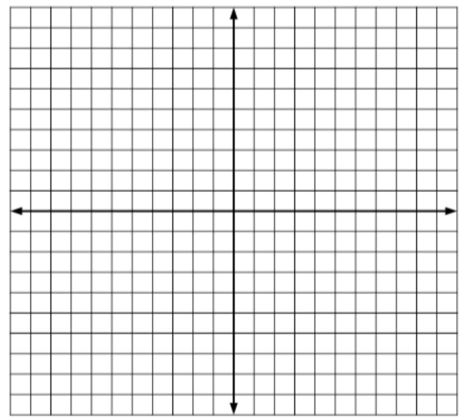


#3-4, graph each function. Then, identify the domain and range.

3. $f(x) = \sqrt{\frac{1}{3}x}$



4. $g(x) = -2\sqrt[3]{x}$



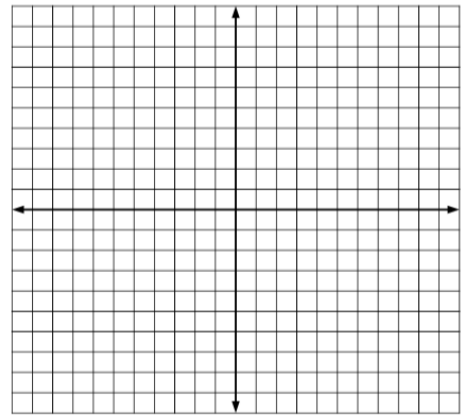
## Core Concept

| Transformation | $f(x)$ Notation | Examples | |
|---|---|---|---|
| **Horizontal Translation**<br>Graph shifts left or right. | $f(x-h)$ | $g(x)=\sqrt{x-2}$<br>$g(x)=\sqrt{x+3}$ | 2 units right<br>3 units left |
| **Vertical Translation**<br>Graph shifts up or down. | $f(x)+k$ | $g(x)=\sqrt{x}+7$<br>$g(x)=\sqrt{x}-1$ | 7 units up<br>1 unit down |
| **Reflection**<br>Graph flips over $x$- or $y$-axis. | $f(-x)$<br>$-f(x)$ | $g(x)=\sqrt{-x}$<br>$g(x)=-\sqrt{x}$ | in the $y$-axis<br>in the $x$-axis |
| **Horizontal Stretch or Shrink**<br>Graph stretches away from or shrinks toward $y$-axis. | $f(ax)$ | $g(x)=\sqrt{3x}$<br>$g(x)=\sqrt{\frac{1}{2}x}$ | shrink by a factor of $\frac{1}{3}$<br>stretch by a factor of 2 |
| **Vertical Stretch or Shrink**<br>Graph stretches away from or shrinks toward $x$-axis. | $a \cdot f(x)$ | $g(x)=4\sqrt{x}$<br>$g(x)=\frac{1}{5}\sqrt{x}$ | stretch by a factor of 4<br>shrink by a factor of $\frac{1}{5}$ |

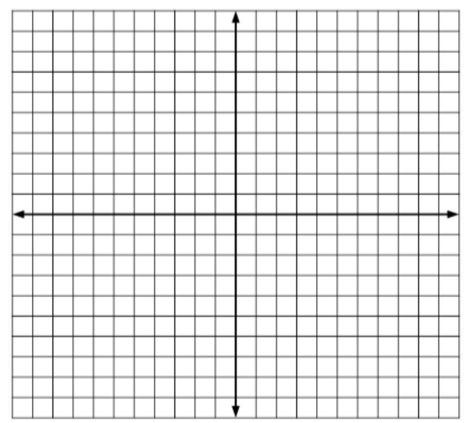
*****CONCEPT 2: TRANSFORMING RADICAL FUNCTIONS*****

5. Describe the transformation of *f* represented by *g*. Then graph each function.

a) $f(x) = \sqrt{x},\ g(x) = \sqrt{x-3} + 4$

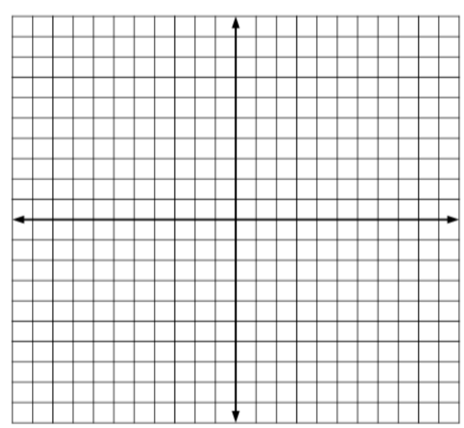


b) $f(x) = \sqrt[3]{x},\ g(x) = \sqrt[3]{-8x}$

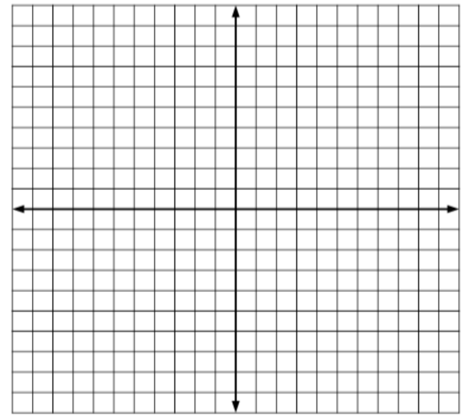


6. Describe the transformation of *f* represented by *g*. Then graph each function.

a) $f(x) = \sqrt{x},\ g(x) = \sqrt{x+2} - 3$



b) $f(x) = \sqrt[3]{x},\ g(x) = -\sqrt[3]{2x}$



*****CONCEPT 3: WRITING A TRANSFORMED RADICAL FUNCTION*****

7. Let the graph be a horizontal shrink by a factor of $\frac{1}{6}$ followed by a translation 3 units to the left of the graph of $f(x) = \sqrt[3]{x}$ . Write the rule for $g$.

8. Let the graph be a horizontal stretch by a factor of 3 followed by a translation 6 units to the right of the graph of $f(x) = \sqrt[3]{x}$ . Write the rule for $g$.

**5.4: Solving Radical Equations and Inequalities (pg. 262 - 265)**

## Core Concept

### Solving Radical Equations

To solve a radical equation, follow these steps:

**Step 1** Isolate the radical on one side of the equation, if necessary.

**Step 2** Raise each side of the equation to the same exponent to eliminate the radical and obtain a linear, quadratic, or other polynomial equation.

**Step 3** Solve the resulting equation using techniques you learned in previous chapters. Check your solution.

*****CONCEPT 1: SOLVING RADICAL EQUATIONS*****

1. Solve (a) $2\sqrt{x+1} = 4$ b) $\sqrt[3]{2x-9} - 1 = 2.$

2. Solve the equations.

a) $2\sqrt{x+2} = 8$ b) $\sqrt[3]{2x-5} - 2 = 3$

*****CONCEPT 2: SOLVING AN EQUATION WITH AN EXTRANEOUS SOLUTION*****

Extraneous solution: a solution that emerges from the solving of a problem but is not **valid.**

3. Solve $x + 1 = \sqrt{7x+15}$ 4. Solve $\sqrt{2x+7} = x - 4$

### *****CONCEPT 3: SOLVING AN EQUATION WITH TWO RADICALS*****

5. Solve $\sqrt{x+2}+1=\sqrt{3-x}$

6. $\sqrt{x+6}-2=\sqrt{x-2}$

### *****CONCEPT 4: SOLVING AN EQUATION WITH A RATIONAL EXPONENT*****

7. Solve $(2x)^{\frac{3}{4}}+2=10$

8. $(3x)^{\frac{2}{3}}-2=34$

### *****CONCEPT 5: SOLVING AN EQUATION WITH A RATIONAL EXPONENT*****

10. Solve $(x+30)^{\frac{1}{2}}=x$

11. $(x+12)^{\frac{1}{2}}=x$

*****CONCEPT 6: SOLVING A RADICAL INEQUALITY*****

12. Solve $3\sqrt{x-1} \leq 12$

13. Solve $3\sqrt{x} - 4 \leq 8$

**5.6: Inverse Functions (pg. 276 - 280)**

Inverse Functions: Functions that *undo* each other. The inverse function is also a reflection on the line $y = x$

*****CONCEPT 1: WRITING A FORMULA FOR THE INPUT OF A FUNCTION*****

1. Let $f(x) = 2x + 3$

a) Solve $y = f(x)$ for *x*

b) Find the input when the output is -7

2. Let $f(x) = 3x - 5$

a) Solve $y = f(x)$ for *x*

b) Find the input when the output is -11

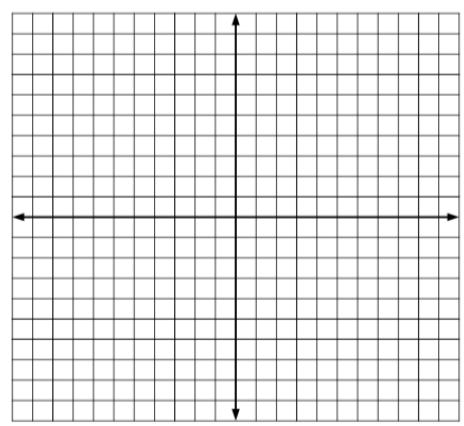
**CONCEPT 2: FINDING THE INVERSE OF A LINEAR FUNCTION**

3. Find the inverse of $f(x) = 3x - 1$

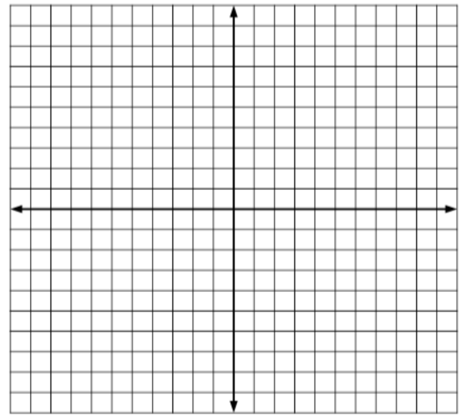
4. Find the inverse of $f(x) = 2x + 8$

**CONCEPT 3: FINDING THE INVERSE OF A QUADRATIC FUNCTION**

5. Find the inverse of $f(x) = x^2$, $x \geq 0$. Then graph the function and its inverse.
(If for the doman of *f* were restricted to $x \geq 0$, then the inverse range would be $y \geq 0$)



6. Find the inverse of $f(x) = x^2 - 2, x \leq 0$. Then graph the function and its inverse.
(If for the doman of *f* were restricted to $x \leq 0$, the inverse would be $y \leq 0$ $or$ $g(x) = -\sqrt{x}$ )

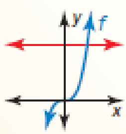


## Core Concept

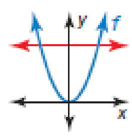
**Horizontal Line Test**

The inverse of a function $f$ is also a function if and only if no horizontal line intersects the graph of $f$ more than once.

**Inverse is a function**

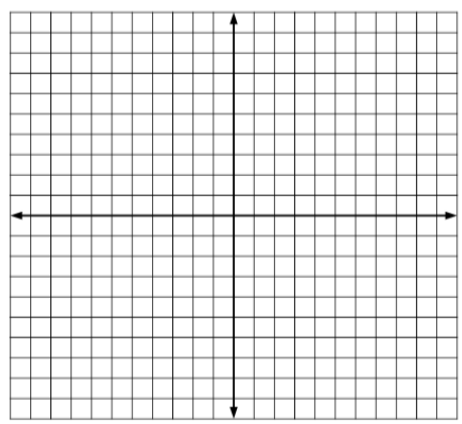



**Inverse is not a function**

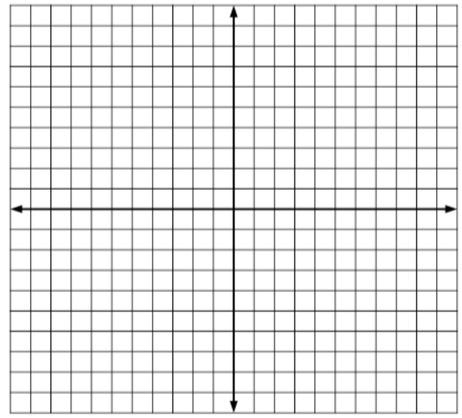



*****CONCEPT 4: FINDING THE INVERSE OF A CUBIC FUNCTION*****

10. Consider the function $f(x) = 2x^3 + 1$. Determine whether the inverse of $f$ is a function. Then find the inverse.

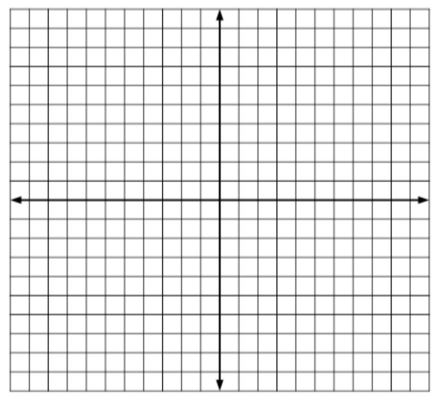


11. Consider the function $f(x) = 3x^3 - 2$. Determine whether the inverse of $f$ is a function. Then find the inverse.

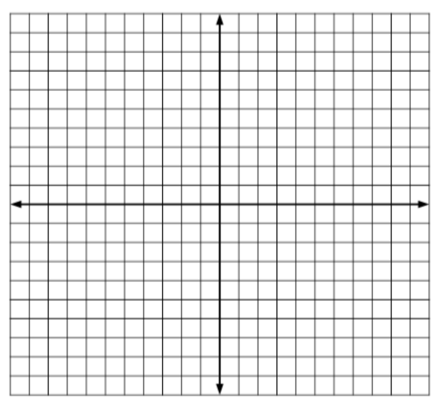


*****CONCEPT 5: FINDING THE INVERSE OF A RADICAL FUNCTION*****

12. Consider the function $f(x) = 2\sqrt{x-3}$. Determine whether the inverse of *f* is a function. Then find the inverse.



13. Consider the function $f(x) = 3\sqrt{x+4}$. Determine whether the inverse of *f* is a function. Then find the inverse.



Inverse functions undo each other. So, when you evaluate a function for a specific input, then evaluate its inverse using the output, you obtain the original input. Let *f* and *g* be inverse functions. So, in general, $f\big(g(x)\big) = x \text{ } and \text{ } g\big(f(x)\big) = x$.

*****CONCEPT 6: VERIFYING FUNCTIONS ARE INVERSES*****

14. Verify that $f(x) = 3x - 1$ and $g(x) = \frac{x+1}{3}$ are inverse functions.

15. Verify that $f(x) = 2x + 4$ and $g(x) = \frac{x-4}{2}$ are inverse functions.