Chapter 4 Polynomial Functions

4.1: Graphing Polynomial Functions (pg. 158-161)

Recall that a monomial is a number, a variable, or the product of a number and one or more variables with whole number exponents. A **polynomial** is a monomial or a sum of monomials. A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, the exponents are all whole numbers, and the coefficients are all real numbers. For this function, a_n is the leading coefficient, n is the degree, and a_0 is the constant term. A polynomial function is in *standard form* when its terms are written in descending order of exponents from left to right.

Common Polynomial Functions			
Degree	**Type**	**Standard Form**	**Example**
0	Constant	$f(x) = a_0$	$f(x) = -14$
1	Linear	$f(x) = a_1x + a_0$	$f(x) = 5x - 7$
2	Quadratic	$f(x) = a_2x^2 + a_1x + a_0$	$f(x) = 2x^2 + x - 9$
3	Cubic	$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$	$f(x) = x^3 - x^2 + 3x$
4	Quartic	$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$	$f(x) = x^4 + 2x - 1$

*****CONCEPT 1: IDENTIFYING POLYNOMIAL FUNCTIONS*****

1. Decide whether each function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

a) $f(x) = 2x^3 + 5x + 8$

b) $g(x) = -0.8x^3 + \sqrt{2}x^4 - 12$

c) $h(x) = -x^2 + 7x^{-1} + 4x$

d) $k(x) = x^2 + 3^x$

Decide whether each function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

2. $f(x) = 7 - 1.6x^2 - 5x$

3. $p(x) = x + 2x^{-2} + 9.5$

4. $q(x) = x^3 - 6x + 3x^4$

*****CONCEPT 2: EVALUATING A POLYNOMIAL FUNCTION*****

5. Evaluate $f(x) = 2x^4 - 8x^2 + 5x - 7$ when $x = 3$.

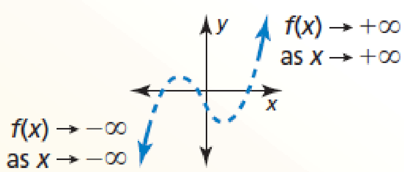
6. Evaluate $f(x) = -2x^4 + 6x^3 - 3x + 11$ when $x = 4$.

The **end behavior** of a function's graph is the behavior of the graph as x approaches positive infinity ($+\infty$) or negative infinity ($-\infty$). For the graph of a polynomial function, the end behavior is determined by the function's degree and the sign of its leading coefficient.

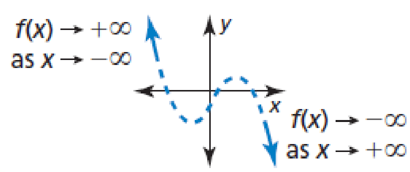
Core Concept

End Behavior of Polynomial Functions

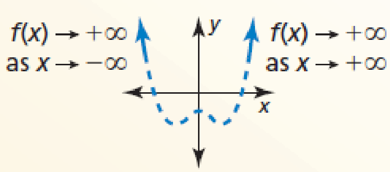
Degree: odd
Leading coefficient: positive



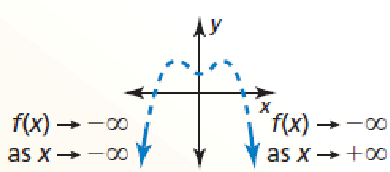
Degree: odd
Leading coefficient: negative



Degree: even
Leading coefficient: positive



Degree: even
Leading coefficient: negative



*****CONCEPT 3: DESCRIBING END BEHAVIOR*****

7. Describe the end behavior of the graph Evaluate $f(x) = -0.5x^4 + 2.5x^2 + x - 1$

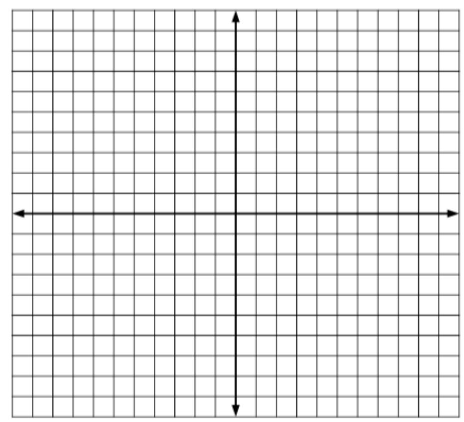
8. Describe the end behavior of the graph Evaluate $f(x) = -0.3x^3 + 1.7x^2 - 4x + 6$

Graphing Polynomial Functions

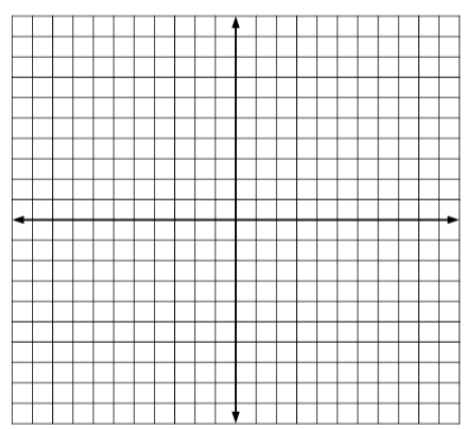
To graph a polynomial function, first plot points to determine the shape of the graph's middle portion. Then connect the points with a smooth continuous curve and use what you know about end behavior to sketch the graph.

*****CONCEPT 4: GRAPHING POLYNOMIAL FUNCTIONS*****

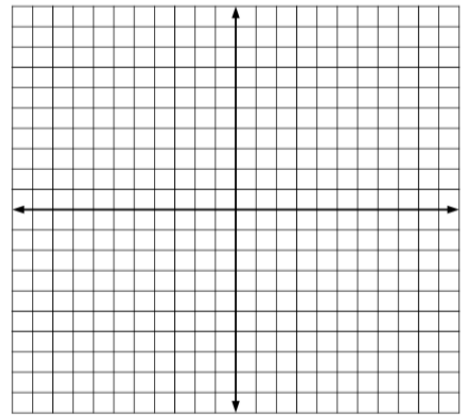
10. Graph (a) $f(x) = -x^3 + x^2 + 3x - 3$



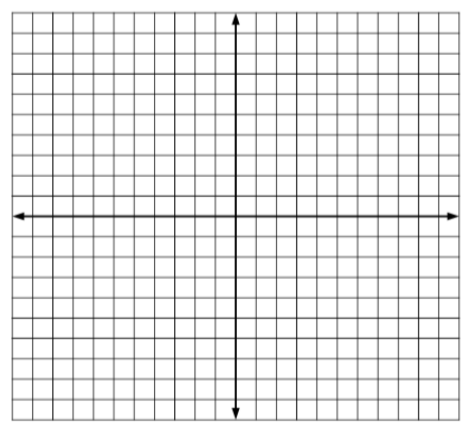
(b) $f(x) = x^4 - x^3 - 4x^2 + 4$



11. Graph $f(x) = x^3 + x^2 - 4x + 2$



12. Graph $f(x) = -x^4 - x^3 + 2x^2 - x - 3$



4.2 Adding, Subtracting, and Multiplying Polynomials (pg. 166-169)

*****CONCEPT 1: ADDING POLYNOMIALS*****

1. Add $3x^3 + 2x^2 - x - 7$ and $x^3 - 10x^2 + 8$

*****CONCEPT 2: SUBTRACTING POLYNOMIALS*****

2.(a) $(8x^3 - 3x^2 - 2x + 9) - (2x^3 + 6x^2 - x + 1)$

(b) Take $3z^2 + z - 4$ from $2z^2 + 3z$

3. $(2x^2 - 6x + 5) + (\ 7x^2 - x - 9)$

4. $(3x^3 - 8x^2 - x - 4) - (\ 5x^3 - x^2 + 17)$

*****CONCEPT 3: MULTIPLYING POLYNOMIALS*****

5. Multiply $y + 5$ and $3y^2 - 2y + 2$

6. Multiply $y - 2$ and $2y^2 - 3y + 5$

*****CONCEPT 4: MULTIPLYING THREE BINOMIALS*****

7. Multiply $x - 1, x + 4$, and $x + 5$

8. Multiply $x + 2, x - 1$, and $x - 3$

Core Concept

Special Product Patterns

Sum and Difference	**Example**
$(a + b)(a - b) = a^2 - b^2$	$(x + 3)(x - 3) = x^2 - 9$
Square of a Binomial	**Example**
$(a + b)^2 = a^2 + 2ab + b^2$	$(y + 4)^2 = y^2 + 8y + 16$
$(a - b)^2 = a^2 - 2ab + b^2$	$(2t - 5)^2 = 4t^2 - 20t + 25$
Cube of a Binomial	**Example**
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	$(z + 3)^3 = z^3 + 9z^2 + 27z + 27$
$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$	$(m - 2)^3 = m^3 - 6m^2 + 12m - 8$

*****CONCEPT 6: USING SPECIAL PRODUCT PATTERNS*****

9. Find the product.

a) $(4n + 5)(4n - 5)$ b) $(9y - 2)^2$ c) $(ab + 4)^3$

10. a) $(2n+7)(2n-7)$ b) $(4y+3)^2$ c) $(ab+2)^3$

Pascal's Triangle

Consider the expansion of the binomial $(a+b)^n$ for whole number values of n. When you arrange the coefficients of the variables in the expansion of $(a+b)^n$, you will see a special pattern called **Pascal's Triangle**. Pascal's Triangle is named after French mathematician Blaise Pascal (1623–1662).

Core Concept

Pascal's Triangle

In Pascal's Triangle, the first and last numbers in each row are 1. Every number other than 1 is the sum of the closest two numbers in the row directly above it. The numbers in Pascal's Triangle are the same numbers that are the coefficients of binomial expansions, as shown in the first six rows.

	n	$(a+b)^n$	Binomial Expansion	Pascal's Triangle
0th row	0	$(a+b)^0 =$	1	1
1st row	1	$(a+b)^1 =$	$1a + 1b$	1 1
2nd row	2	$(a+b)^2 =$	$1a^2 + 2ab + 1b^2$	1 2 1
3rd row	3	$(a+b)^3 =$	$1a^3 + 3a^2b + 3ab^2 + 1b^3$	1 3 3 1
4th row	4	$(a+b)^4 =$	$1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$	1 4 6 4 1
5th row	5	$(a+b)^5 =$	$1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$	1 5 10 10 5 1

*****CONCEPT 7: USING PASCAL'S TRIANGLE TO EXPAND BINOMIALS*****

11. Use Pascal's Triangle to expand (a) $(x-2)^5$ and (b) $(3y+1)^3$

12. Use Pascal's Triangle to expand (a) $(x-3)^4$ and (b) $(2x+4)^3$

4.3: Dividing Polynomials (pg. 174-176)

Dividing Polynomials

- There are two methods for dividing polynomials. You will need to know both
    1. *Long Division*
    2. *Synthetic Division*

1. **LONG DIVISON:**

    When you divide a polynomial $f(x)$ by a nonzero polynomial divisor $d(x)$, you get a quotient polynomial $q(x)$ and a remainder polynomial $r(x)$.

    $$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

*****CONCEPT 1: USING POLYNOMIAL LONG DIVISION*****

1. Divide $2x^4 + 3x^3 + 5x - 1$ by $x^2 + 3x + 2$

2. $(x^3 - x^2 - 2x + 8) \div (\ x - 1)$

3. $(x^3 + 2x^2 - x + 5) \div (x^2 - x + 1)$

2. **SYNTHETIC DIVISION**

- A shortcut for dividing polynomials by binomials of the form *(x - k)*
  →How to use Synthetic Division:

*****CONCEPT 2: USING SYNTHETIC DIVISION*****

4. Divide $-x^3 + 4x^2 + 9$ by $x - 3$

5. Divide $3x^3 - 2x^2 + 2x - 5$ by $x + 1$

Divide using synthetic division

6. $(-x^3 + 3x^2 + x) \div (\,x - 2)$

7. $(3x^3 - 7x^2 + 6x + 8) \div (\,x - 1)$

Core Concept

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

- You can <u>use the Remainder Theorem</u> to tell whether *synthetic division* can be used to evaluate a polynomial function
- Your goal is to *determine the remainder of the function*

*****CONCEPT 3: EVALUATING A POLYNOMIAL*****

8. Use synthetic division to evaluate $f(x) = 5x^3 - x^2 + 13x + 29;\; x = -4$

Use synthetic division to evaluate the function for the indicated value of x.

9. $f(x) = 4x^2 - 10x - 21; x = 5$

10. $f(x) = 4x^3 - 2x^2 - 5x + 11; x = -2$

4.4: Factoring Polynomials (pg. 180-182)

Factoring Polynomials

- You've previously factored *Quadratic Equations* of degree 2
- You can also factor Polynomials with degree greater than 2

**Recall that a *factor* is a number that is multiplied to get a product

*****CONCEPT 1: FINDING A COMMON MONOMIAL FACTOR*****

1. Factor each polynomial completely

a) $x^3 - 4x^2 - 5x$ b) $3x^5 - 48x^3$ c) $5x^4 + 30x^3 + 45x^2$

Factor each polynomial completely

2. $x^3 - 7x^2 + 10x$ 3. $3x^7 - 75x^5$

Core Concept

Special Factoring Patterns

Sum of Two Cubes

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Example

$64x^3 + 1 = (4x)^3 + 1^3$

$= (4x + 1)(16x^2 - 4x + 1)$

Difference of Two Cubes

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Example

$27x^3 - 8 = (3x)^3 - 2^3$

$= (3x - 2)(9x^2 + 6x + 4)$

*****CONCEPT 2: FACTORING THE SUM OR DIFFERENCE OF TWO CUBES*****

4. Factor (a) $3x^3 - 125$ and (b) $16x^5 + 54x^2$ completely

5. Factor (a) $x^3 - 64$ and (b) $-16x^5 - 250x^2$ completely

*****CONCEPT 3: FACTORING BY GROUPING*****

6. Factor $x^3 + 5x^2 - 4x - 20$ completely.

7. Factor $x^3 - 2x^2 - 9x + 18$ completely.

*****CONCEPT 4: FACTORING POLYNOMIALS IN QUADRATIC FORM*****

8. Factor (a) $16x^4 - 81$ and (b) $3x^8 + 15x^3 + 18x^2$ completely.

9. Factor (a) $625x^8 - 256$ and (b) $2x^{13} + 10x^9 + 8x^5$ completely.

Core Concept

The Factor Theorem

A polynomial $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

*****CONCEPT 5: DETERMINING WHETHER A LINEAR BINOMIAL IS A FACTOR*****

10. Determine whether (a) $x-2$ is a factor of $f(x) = x^2 + 2x - 4$ and then (b) $x + 5$ is a factor of $f(x) = 3x^4 + 15x^3 - x^2 + 25$

*****CONCEPT 6: FACTORING A BINOMIAL*****

11. Show that $x + 3$ is a factor of $f(x) = x^4 + 3x^3 - x - 3$. Then factor completely.

12. Show that $x - 2$ is a factor of $f(x) = x^4 - 2x^3 + x - 2$. Then factor completely.

4.5 Solving Polynomial Equations (pg. 190-193)

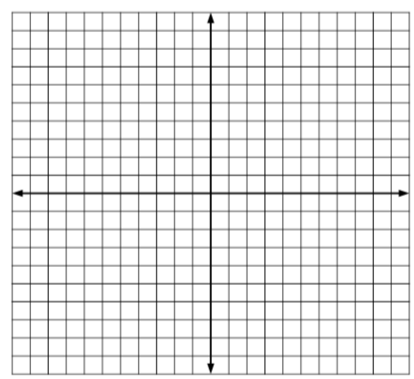
- Repeated Solution: A solution that appears more than once

*****CONCEPT 1: SOLVING A POLYNOMIAL BY FACTORING*****

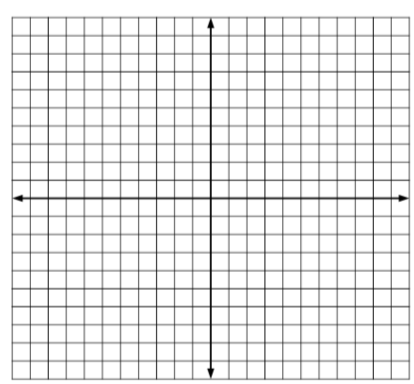
1. Solve $2x^3 - 12x^2 + 18x = 0$

*****CONCEPT 2: FINDING ZEROS OF A POLYNOMIAL*****

2. Find the zeros of $f(x) = -2x^4 + 16x^2 - 32$. Then sketch a graph of the function.



3. Find the zeros of $f(x) = x^4 - 18x^2 + 81$. Then sketch a graph of the function.



Core Concept

The Rational Root Theorem

If $f(x) = a_n x^n + \cdots + a_1 x + a_0$ has *integer* coefficients, then every rational solution of $f(x) = 0$ has the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

- The Rational Root Theorem lists *only possible* solutions
  - To find the ACTUAL solutions, you must test these values via *Synthetic Division*

*****CONCEPT 3: USING THE RATIONAL ROOT THEOREM*****

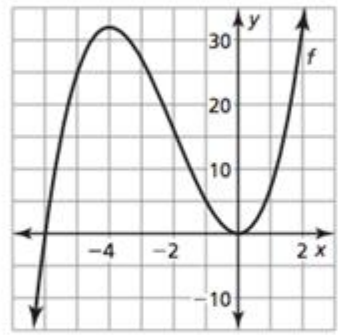
4. Find all real solutions of $x^3 - 8x^2 + 11x + 20 = 0$

5. Find all real solutions of $\mathrm{x}^3 - 2\mathrm{x}^2 - 5\mathrm{x} - 6 = 0$

*****CONCEPT 4: USING ZEROS TO WRITE A POLYNOMIAL FUNCTION*****

6. Use the information in the graph to answer the questions.

a. What are the real zeros of the function f?



b. Write an equation of the cubic function in factored form.

Core Concept

The Irrational Conjugates Theorem

Let f be a polynomial function with rational coefficients, and let a and b be rational numbers such that $\sqrt{b}$ is irrational. If $a + \sqrt{b}$ is a zero of f, then $a - \sqrt{b}$ is also a zero of f.

7. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 3 and $2 + \sqrt{5}$.

8. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 2 and $3 + \sqrt{7}$.

4.6: The Fundamental Theorem of Algebra

Core Concept

The Fundamental Theorem of Algebra

Theorem If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has at least one solution in the set of complex numbers.

Corollary If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has exactly n solutions provided each solution repeated twice is counted as two solutions, each solution repeated three times is counted as three solutions, and so on.

The table shows several polynomial equations and their solutions, including repeated solutions. Notice that for the last equation, the repeated solution $x = -1$ is counted twice.

Equation	Degree	Solution(s)	Number of solutions
$2x - 1 = 0$	1	$\frac{1}{2}$	1
$x^2 - 2 = 0$	2	$\pm\sqrt{2}$	2
$x^3 - 8 = 0$	3	$2, -1 \pm i\sqrt{3}$	3
$x^3 + x^2 - x - 1 = 0$	3	$-1, -1, 1$	3

In the table, note the relationship between the degree of the polynomial $f(x)$ and the number of solutions of $f(x) = 0$. This relationship is generalized by the *Fundamental Theorem of Algebra*, first proven by German mathematician Carl Friedrich Gauss (1777–1855).

*****CONCEPT 1: FINDING THE NUMBER OF SOLUTIONS OR ZEROS*****

1. a) $f(x) = x^3 + 3x^2 + 16x + 48 = 0$ b) $f(x) = x^4 + 6x^3 + 12x^2 + 8x = 0$

*****CONCEPT 2: FINDING THE ZEROS OF A FUNCTION*****

2. Find all zeros of $f(x) = x^5 + x^3 - 2x^2 - 12x - 8$

3. Find all zeroes of $f(\mathrm{x}) = \mathrm{x}^4 - \mathrm{x}^3 + 7\mathrm{x}^2 + 16\mathrm{x} + 12$

*****CONCEPT 3: USING ZEROS TO WRITE A POLYNOMIAL FUNCTION*****

4. Write a polynomial function *f* of least degree that has rational coefficients, a lead coefficient of 1, and zeros 2 and $3 + i$.

5. Write a polynomial function *f* of least degree that has rational coefficients, a leading coefficient of 1, and zeros 5 and $1 + i$.

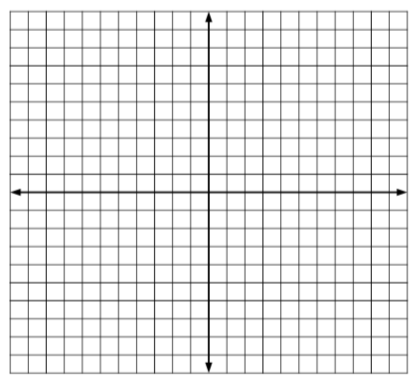
4.7: Transformations of Polynomials (pg. 206-208)

Core Concept

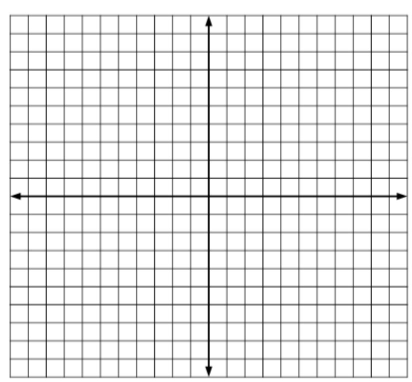
Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x-h)$	$g(x)=(x-5)^4$ $g(x)=(x+2)^4$	5 units right 2 units left
Vertical Translation Graph shifts up or down.	$f(x)+k$	$g(x)=x^4+1$ $g(x)=x^4-4$	1 unit up 4 units down
Reflection Graph flips over x- or y-axis.	$f(-x)$ $-f(x)$	$g(x)=(-x)^4=x^4$ $g(x)=-x^4$	over y-axis over x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis.	$f(ax)$	$g(x)=(2x)^4$ $g(x)=\left(\frac{1}{2}x\right)^4$	shrink by a factor of $\frac{1}{2}$ stretch by a factor of 2
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis.	$a \cdot f(x)$	$g(x)=8x^4$ $g(x)=\frac{1}{4}x^4$	stretch by a factor of 8 shrink by a factor of $\frac{1}{4}$

*****CONCEPT 1: TRANSLATING A POLYNOMIAL FUNCTION*****

1. Describe the transformation of $f(x)=x^3$ represented by the $g(x)=(x+5)^3+2$. Then graph each function.



2. Describe the transformation of $f(x)=x^3$ represented by the $g(x)=2(x)^3-6$. Then graph each function.



*****CONCEPT 2: WRITING A TRANSFORMED POLYNOMIAL FUNCTION*****

3. Let the graph of g be a vertical stretch by a factor of 2, followed by a translation 3 units up of the graph of $f(x) = x^4 - 2x^2$. Write a rule for g.

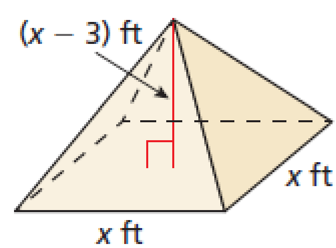
4. Let the graph of g be a vertical stretch by a factor of $\frac{1}{2}$, followed by a translation 4 units down of the graph of $f(x) = x^5 - 2x^2$. Write a rule for g.

5. Let the graph of g be a horizontal stretch by a factor of 2, followed by a translation 3 units to the right of the graph of $f(x) = 8x^3 + 3$. Write a rule for g.

6. Let the graph of g be a vertical stretch by a factor of 2, followed by a translation 4 units to the right of the graph of $f(x) = 2x^3 + x + 1$. Write a rule for g.

*****CONCEPT 3: MODELING WITH MATHEMATICS*****

7. The function $V(x) = \frac{1}{3}x^3 - x^2$ represents the volume (in cubic feet) of the square pyramid shown. The function $W(x) = V(3x)$ represents the volume (in cubic feet) when *x* is measured in yards. Write a rule for W. Find and interpret $W(10)$.



8. The function $V(x) = x^3$ represents the volume (in cubic feet) of a cube with side length *x*. The function $W(x) = V\left(\frac{1}{12}x\right)$ represents the volume (in cubic feet) when x is measured in inches. Write a rule for W. Find and interpret $W(96)$.

4.8: Analyzing Graphs of Polynomials (pg. 212-215)

Concept Summary

Zeros, Factors, Solutions, and Intercepts

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial function. The following statements are equivalent.

Zero: k is a zero of the polynomial function f.

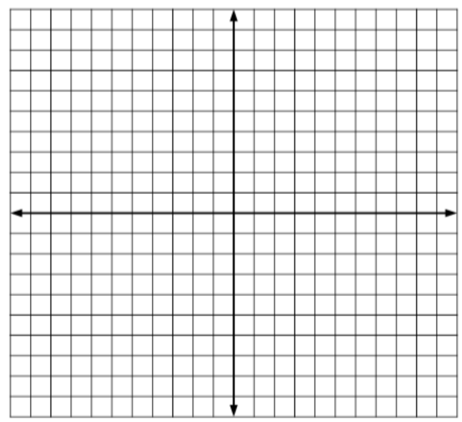
Factor: $x - k$ is a factor of the polynomial $f(x)$.

Solution: k is a solution (or root) of the polynomial equation $f(x) = 0$.

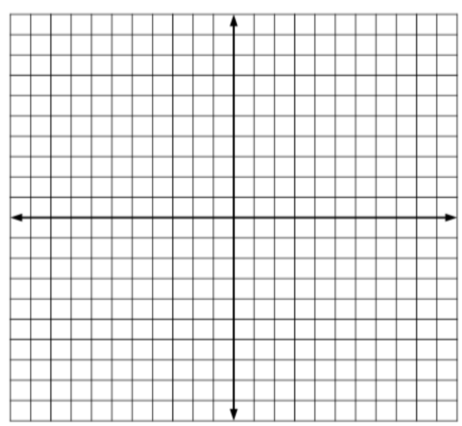
***x*-Intercept:** If k is a real number, then k is an x-intercept of the graph of the polynomial function f. The graph of f passes through $(k, 0)$.

*****CONCEPT 1: USING X-INTERCEPTS TO GRAPH A POLYNOMIAL*****

1. Graph the function $f(x) = \frac{1}{6}(x+3)(x-2)^2$



2. Graph the function $f(x) = \frac{1}{2}(x+1)(x-4)^2$

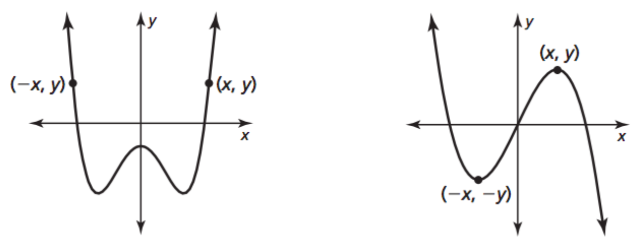


Even and Odd Functions

A function f is an **even function** when $f(-x) = f(x)$ for all x in its domain. The graph of an even function is *symmetric about the y-axis*.

A function f is an **odd function** when $f(-x) = -f(x)$ for all x in its domain. The graph of an odd function is *symmetric about the origin*. One way to recognize a graph that is symmetric about the origin is that it looks the same after a $180°$ rotation about the origin.

Even Function **Odd Function**



For an even function, if (x, y) is on the graph, then $(-x, y)$ is also on the graph.

For an odd function, if (x, y) is on the graph, then $(-x, -y)$ is also on the graph.

*****CONCEPT 2: IDENTIFYING EVEN AND ODD FUNCTIONS*****

3. Determine whether each function is *even, odd, or neither.*

a) $f(x) = x^3 - 7x$ b) $g(x) = x^4 + x^2 - 1$ c) $h(x) = x^3 + 2$

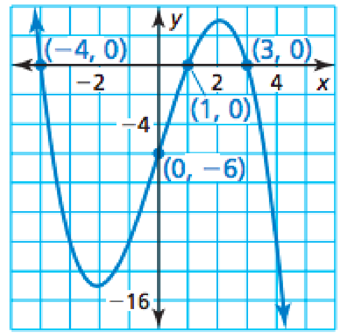
4. Determine whether the function is *even, odd,* or *neither.*

a) $f(x) = -x^2 + 5$ b) $g(x) = x^4 - 5x^3 - 7$ c) $h(x) = 2x^5 - 12x$

4.9: Modeling with Polynomial Functions (pg. 220-222)

*****CONCEPT 1: WRITING A CUBIC FUNCTION*****

1. Write the cubic function whose graph is shown.



2. Write the cubic function whose graph is shown.